Probabilities of concurrent extremes

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1. Introduction

Definition 1. The process $\{\eta(x) \colon x \in \mathcal{X}\}$ is said to be max-stable if for all $n \geq 1$ there exist continuous normalizing functions $a_n(\cdot) > 0$ and $b_n(\cdot) \in \mathbb{R}$ such that

$$\left\{ \frac{\max_{i=1,\dots,n} \eta_i(x) - b_n(x)}{a_n(x)} \colon x \in \mathcal{X} \right\} \stackrel{\mathrm{d}}{=} \{\eta(x) \colon x \in \mathcal{X}\}$$

where $\eta_1, \dots, \eta_n$ are independent copies of the process $\{\eta(x) \colon x \in \mathcal{X}\}$.

Remark. Throughout this talk we will assume that $\mathcal{X} \subset \mathbb{R}^d$, $d \geq 1$, is compact and that all stochastic processes have continuous sample paths.

Theorem 1. *(de Haan and Fereira, 2006)*
Let $\{X_i(x)\colon x \in \mathcal{X}, i \geq 1\}$ be a sequence of independent copies of a stochastic process $\{X(x)\colon x \in \mathcal{X}\}$. If there exist sequences of normalizing functions $\{c_n(x) > 0\colon x \in \mathcal{X}, n \geq 1\}$ and $\{d_n(x) \in \mathbb{R}\colon x \in \mathcal{X}, n \geq 1\}$ then, provided the limiting process is non degenerate,

$$\left\{ \frac{\max_{i=1,\dots,n} X_i(x) - d_n(x)}{c_n(x)} : x \in \mathcal{X} \right\} \xrightarrow{\mathrm{d}} \{\eta(x)\colon x \in \mathcal{X}\},$$

as $n \to \infty$, it has to be a max-stable process.

- The finite dimensional distributions are multivariate extreme value distributions and, in particular, $\eta(x) \sim$ GEV, $x \in \mathcal{X}$.
- If $\{\eta(x)\colon x \in \mathcal{X}\}$ has unit Fréchet margins, i.e., $\Pr\{\eta(x) \leq z\} = \exp(-1/z)$, $z > 0$, we say that it is a simple max-stable process.

Theorem 2. *(de Haan, 1984; Penrose, 1992)*
Any simple max-stable process $\{\eta(x) \colon x \in \mathcal{X}\}$ *can be represented as follows*

$$\{\eta(x) \colon x \in \mathcal{X}\} \stackrel{\mathrm{d}}{=} \left\{ \max_{\varphi \in \Phi} \varphi(x) \colon x \in \mathcal{X} \right\},$$

where Φ *is a Poisson point process on* $\mathbb{C}_0 = \mathbb{C}\{\mathcal{X}, [0,\infty)\} \setminus \{0\}$ *with intensity measure*

$$\Lambda(A) = \int_0^\infty \Pr(\zeta Y \in A) \zeta^{-2} d\zeta, \qquad A \subset \mathbb{C}_0 \text{ Borel set},$$

and where $\{Y(x) \colon x \in \mathcal{X}\}$ *is a non negative stochastic process such that* $\mathbb{E}\{Y(x)\} = 1$, $x \in \mathcal{X}$ *and* $\mathbb{E}\left\{\sup_{x \in \mathcal{X}} Y(x)\right\} < \infty$.

Popcorn time. . .

Example 1. (Brown and Resnick, 1977; Kabluchko et al., 2009)
The Brown–Resnick model consists in taking

$$\{Y(x) \colon x \in \mathcal{X}\} \stackrel{\mathrm{d}}{=} \{\exp\{\varepsilon(x) - \gamma(x)\} : x \in \mathcal{X}\},$$

where $\{\varepsilon(x) \colon x \in \mathcal{X}\}$ is a centered Gaussian process with stationary increments and semi variogram γ.

Example 2. (Davison et al., 2012; Opitz, 2013)
The extremal–t model consists in taking

$$\{Y(x) \colon x \in \mathcal{X}\} \stackrel{\mathrm{d}}{=} \{c_\nu \max\{0, \varepsilon(x)\}^\nu \colon x \in \mathcal{X}\},$$

where $\nu \geq 1$ and $\{\varepsilon(x) \colon x \in \mathcal{X}\}$ is a standard Gaussian process with correlation function ρ and c_ν is a normalizing constant.

Hitting scenario

- The way the atoms of Φ contribute to $\{\eta(x) \colon x \in \mathcal{X}\}$ at locations $x_1, \dots, x_k$ defines a hitting scenario.

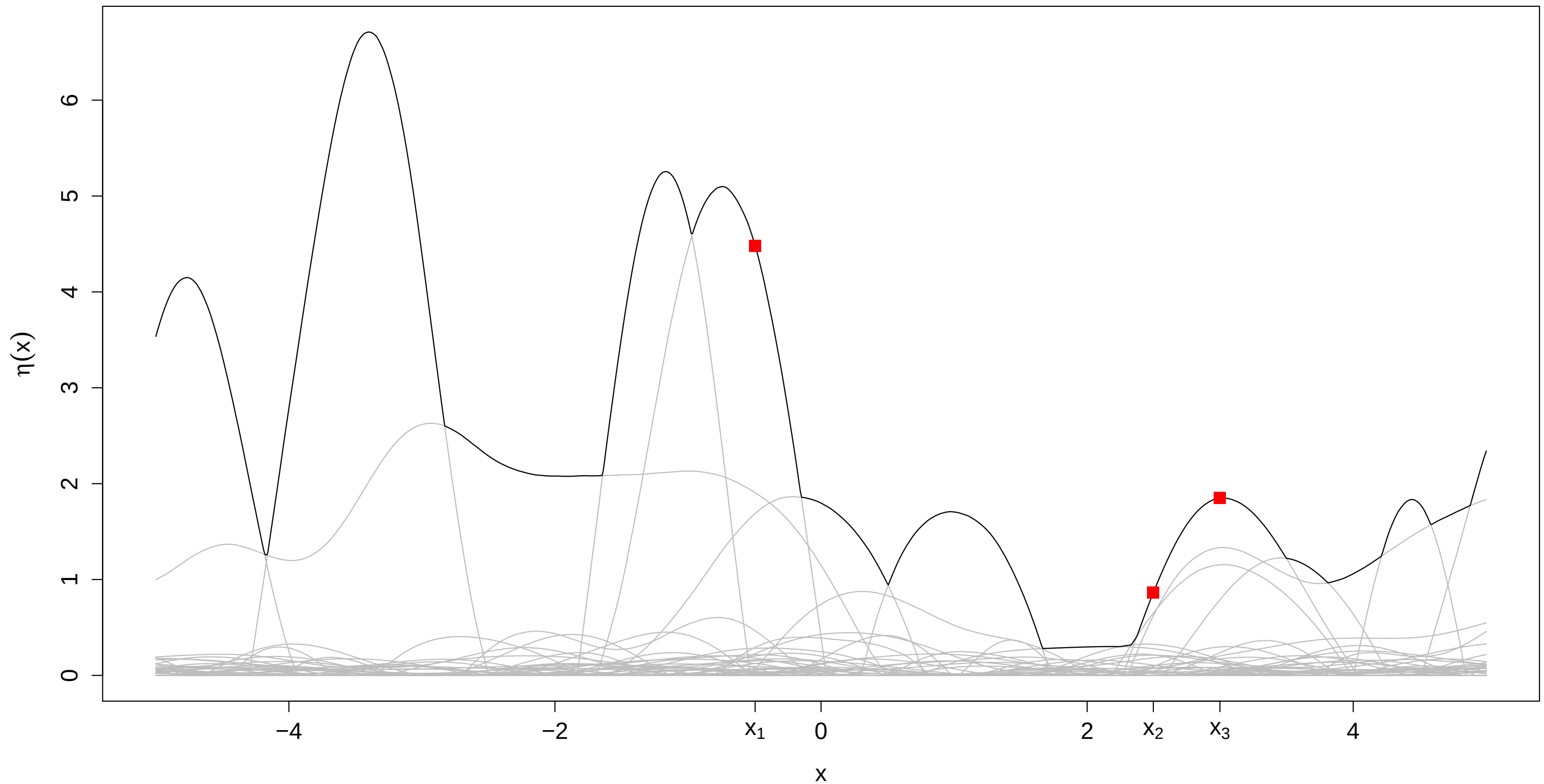

Figure 1: *Illustration of the notion of a hitting scenario. Here the hitting scenario is* $\tau = \{\{x_1\}, \{x_2, x_3\}\}$.

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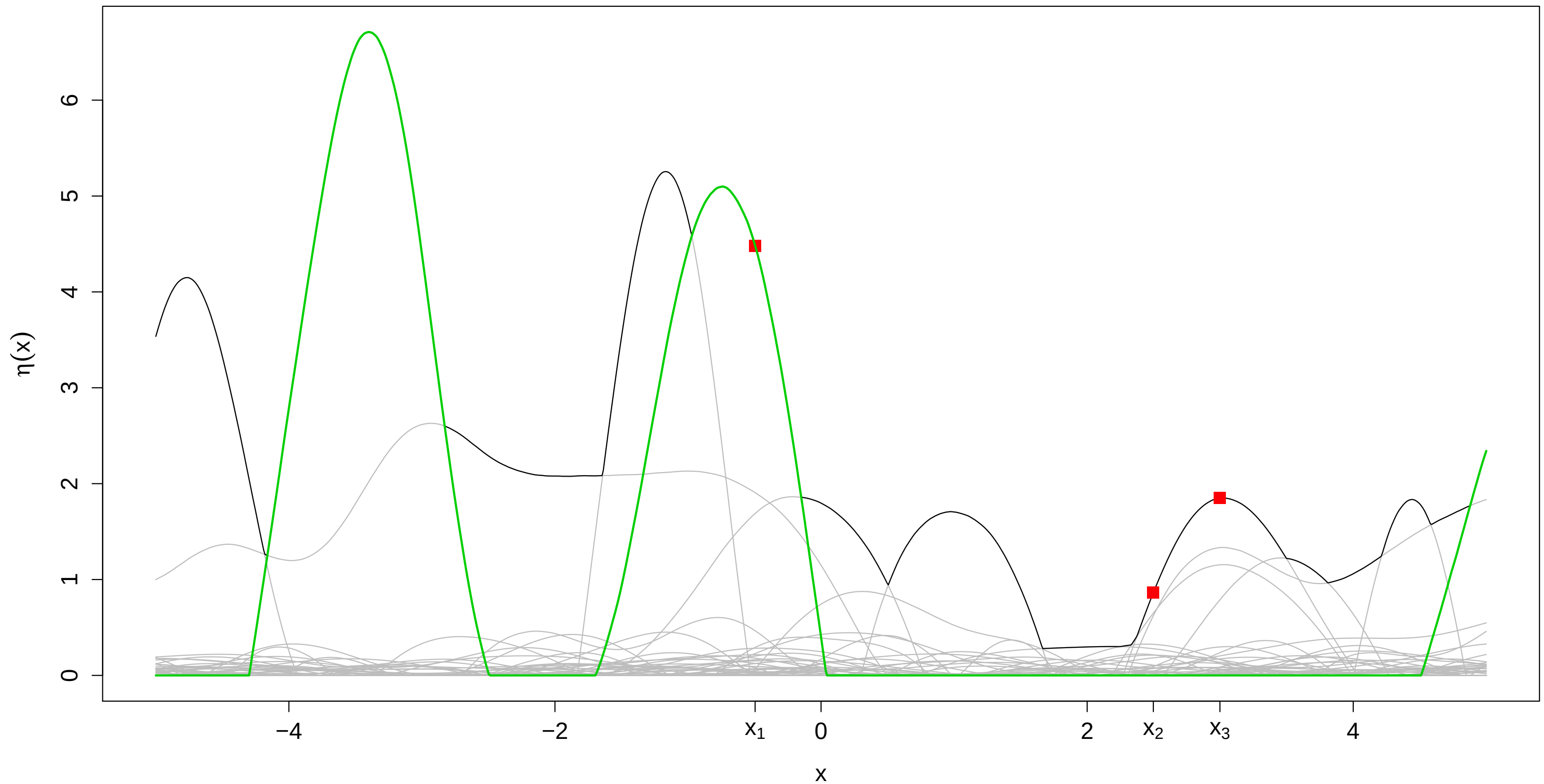

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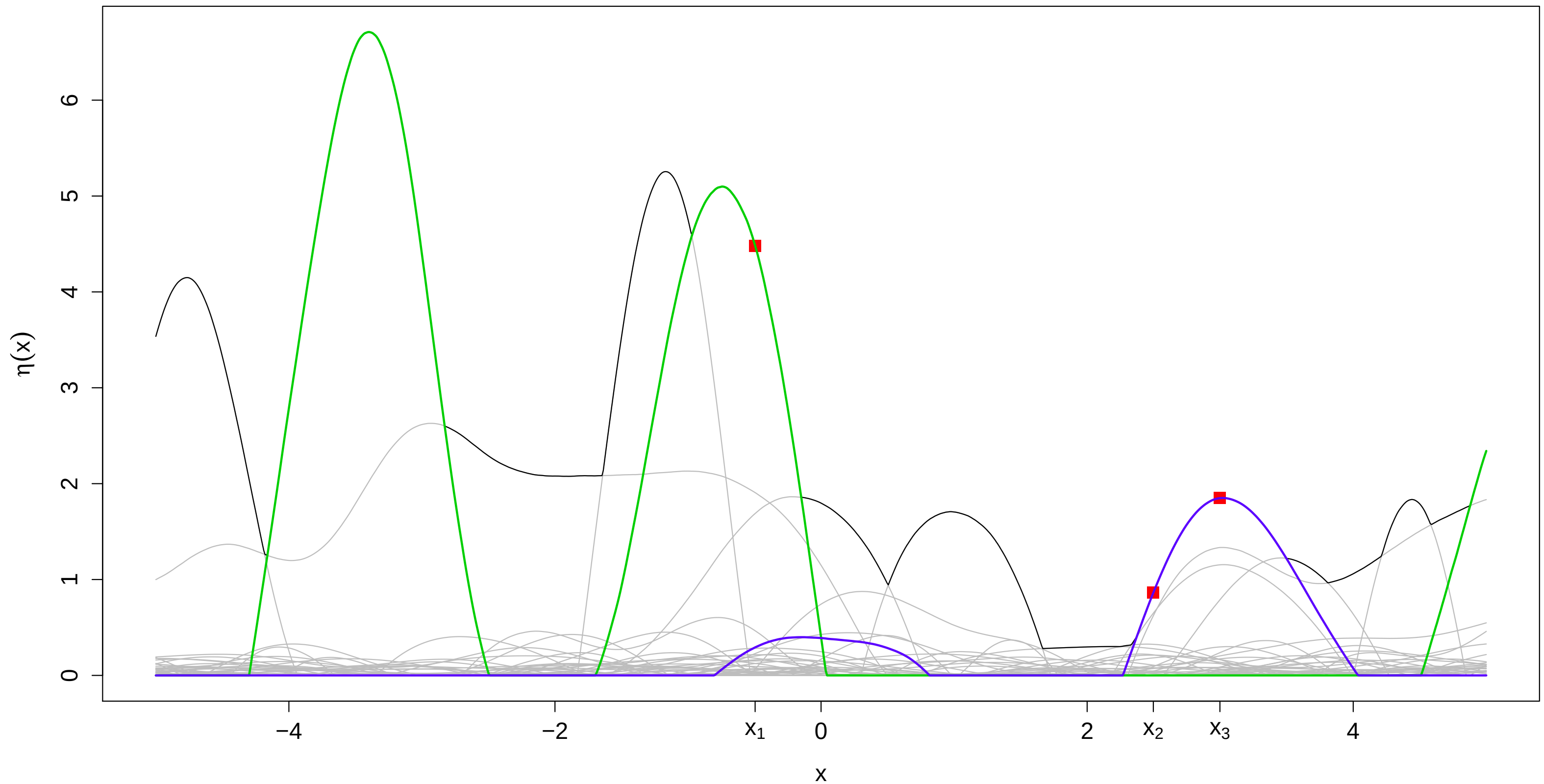

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- Let $\mathscr{P}_k$ the set of all possible partitions of $\{x_1, \ldots, x_k\}$.
- The density of $\{\eta(x_1), \ldots, \eta(x_k)\}$ is

$$f(z) = \exp\{-V(z)\} \sum_{\tau \in \mathscr{P}_k} \prod_{j=1}^{|\tau|} \int_{(0, z_{\tau_{-j}})} \lambda(z_{\tau_j}, u_j) \mathrm{d}u_j.$$

Example 3. (Dombry et al., 2013; Dombry and Éyi-Minko, 2013)
For the Brown–Resnick model, λ is (related to) a multivariate log-normal distribution.

Example 4. (Ribatet, 2013)
For the Extremal-t, λ is (related to) a multivariate Student distribution.

Dependence measure

- The extremal coefficients summarize the dependence of max-stable random vectors

$$\underset{\text{dependence}}{1} \leq \theta = -z \log \Pr\{\eta(x_j) \leq z \colon j = 1, \ldots, k\} \leq \underset{\text{independence}}{k}.$$

- In a spatial context, it is more convenient to plot the extremal coefficient function

$$\begin{aligned} \theta \colon \mathcal{X} &\longrightarrow [1, 2] \\ h &\longmapsto \theta(o, h), \end{aligned}$$

  which plays a similar role to the semivariogram in conventional geostatistics.
- Usually isotropy is assumed, i.e., $\theta(h) = \theta(\|h\|)$.

Spatial dependence

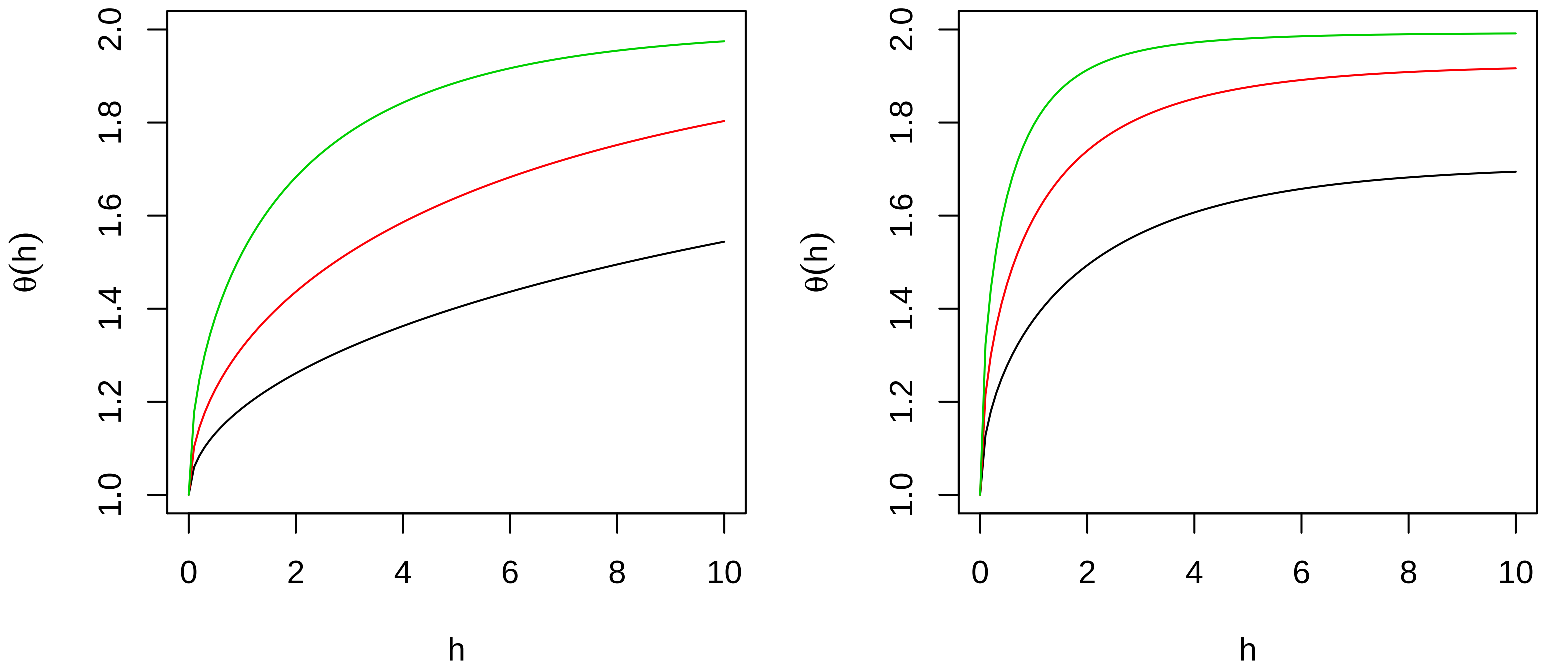

Figure 2: *Plot of various extremal coefficient functions. Left: Brown–Resnick model. Right: Extremal-t model.*

- We get a rough picture of how dependence decreases.
- What's the difference between $\theta(h) = 1.2$ and $\theta(h) = 1.4$???

2. Probabilities of concurrent extremes

Definition 2. Let $\{X_i(x)\colon x \in \mathcal{X}, i = 1, \ldots, n\}$ be independent copies of a stochastic process $\{X(x)\colon x \in \mathcal{X}\}$—with continuous margins. Extremes are said sample concurrent at locations $(x_1, \ldots, x_k)$, $k \geq 2$, if there exists $\ell \in \{1, \ldots, n\}$ such that

$$\max_{i=1,\ldots,n} X_i(x_j) = X_\ell(x_j), \qquad \text{for all } j = 1, \ldots, k.$$

The associated sample concurrence probability is

$$p_n(x_1, \ldots, x_k) = \Pr\{\text{sample concurrence occurs at } (x_1, \ldots, x_k)\}.$$

□ It is not difficult to show that

$$p_n(x_1, \ldots, x_k) = n\mathbb{E}\left[F\left\{X(x_1), \ldots, X(x_k)\right\}^{n-1}\right].$$

Definition 3. Let $\{\eta(x) \colon x \in \mathcal{X}\}$ be a (simple) max-stable process with spectral characterization $\eta(\cdot) = \max_{\varphi \in \Phi} \varphi(\cdot)$. Extremes are said extremal concurrent at location $(x_1, \dots, x_k)$, $k \geq 2$, if there exists $\ell \geq 1$ such that

$$\eta(x_j) = \varphi_\ell(x_j), \qquad \text{for all } j = 1, \dots, k.$$

The associated extremal concurrence probability is

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Remark. Extremal concurrence for $(x_1, \ldots, x_k) \iff$ the hitting scenario is $\tau = \{x_1, \ldots, x_k\}$.

Proposition 1. *Let* $\{X(x)\colon x \in \mathcal{X}\}$ *be a stochastic process that belongs to the max-domain of attraction of some max-stable process* $\{\eta(x)\colon x \in \mathcal{X}\}$. *Then for all* $x_1, \ldots, x_k \in \mathcal{X}$, $k \geq 2$,

$$p_n(x_1, \ldots, x_k) \longrightarrow p(x_1, \ldots, x_k), \qquad n \to \infty.$$

Remark. Actually we can show a bit more than the above:

the sample hitting scenario converges to the extremal one.

Theorem 3. *Let $\{Y(x) \colon x \in \mathcal{X}\}$ and $\{\tilde{Y}(x) \colon x \in \mathcal{X}\}$ be two independent copies of the process appearing in the spectral characterization. Then for all $x_1, \ldots, x_k \in \mathcal{X}$, $k \geq 2$,*

$$p(x_1, \ldots, x_n) = \mathbb{E}_Y \left(\left[\mathbb{E}_{\tilde{Y}} \left\{ \max_{j=1,\ldots,k} \frac{\tilde{Y}(x_j)}{Y(x_j)} \right\} \right]^{-1} \right),$$

or equivalently in terms of the V function

$$p(x_1, \ldots, x_n) = \mathbb{E} \left[\frac{1}{V\{Y(x_1), \ldots, Y(x_k)\}} \right].$$

Remark. Typically closed forms won't be available but the above equation suggests a (simple) Monte Carlo estimator.

Proposition 2. *Let $\{\tilde\eta(x) \colon x \in \mathcal{X}\}$ be an independent copy of $\{\eta(x) \colon x \in \mathcal{X}\}$. Then for all $x_1, \dots, x_k \in \mathcal{X}$, $k \geq 2$,*

$$p(x_1, \dots, x_k) = \sum_{r=1}^{k} (-1)^r \sum_{\substack{J \subseteq \{1,\dots,k\} \\ |J|=r}} \mathbb{E}_{\tilde\eta} \left(\log \Pr_{\eta} \left[\{\eta(x_j) \leq \tilde\eta(x_j), j \in J\} \right] \right),$$

or equivalently in terms of the V function

$$p(x_1, \dots, x_k) = \sum_{r=1}^{k} (-1)^{r+1} \sum_{\substack{J \subseteq \{1,\dots,k\} \\ |J|=r}} \mathbb{E} \left[V\{\eta(x_j) \colon j \in J\} \right].$$

Remark. As expected the extremal concurrence probability does not depend on a specific spectral representation but only on the distribution of $\{\eta(x) \colon x \in \mathcal{X}\}$.

Relationship with extremal coefficients

Proposition 3. *Let* $\{\tilde{\eta}_i(x) \colon x \in \mathcal{X}, i \geq 1\}$ *be a sequence of independent copies of a simple max-stable process* $\{\eta(x) \colon x \in \mathcal{X}\}$*. Then the process*

$$\{\xi(x) \colon x \in \mathcal{X}\} = \left\{ \max_{i=1,\ldots,n} \zeta_i \frac{Y_i(x)}{\tilde{\eta}_i(x)} \colon x \in \mathcal{X} \right\}$$

is a simple max-stable process and for all $x_1, \ldots, x_k \in \mathcal{X}$*,* $k \geq 2$*,*

$$p(x_1, \ldots, x_k) = \sum_{r=1}^{k} (-1)^{r+1} \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ |J| = r}} \theta_\xi(x_j \colon j \in J),$$

where $\theta_\xi(x_j \colon j \in J) = -\log \Pr\{\xi(x_j) \leq 1 \colon j \in J\}$*.*

- The above expression simplify a lot when $k = 2$.
- In particular $p(x_1, x_2) = 2 - \theta_\xi(x_1, x_2)$ and we can define an extremal concurrence probability function and gets its properties for free (Schlather and Tawn, 2003; Cooley et al., 2006)!

Proposition 4. *Let $p\colon h \mapsto p(o, h)$ be the extremal concurrence probability function of a stationary max-stable process $\{\eta(x)\colon x \in \mathcal{X}\}$. Then*

i) $h \mapsto p(h)$ *is positive semidefinite,*
ii) $h \mapsto p(h)$ *is not differentiable at the origin unless* $p \equiv 1$.
iii) *If $d \geq 1$ and η is isotropic, then $h \mapsto p(h)$ has at most one jump at the origin and is continuous elsewhere.*
iv) $2 - p(h_1 + h_2) \leq \{2 - p(h_1)\}\{2 - p(h_2)\}$.

Proposition 5. *For all* $x_1, x_2 \in \mathcal{X}$,

$$p(x_1, x_2) = 0 \iff \eta(x_1) \text{ and } \eta(x_2) \text{ are independent},$$
$$p(x_1, x_2) = 1 \iff \eta(x_1) \text{ and } \eta(x_2) \text{ are completely dependent},$$

and $\frac{1}{2}\{2 - \theta(x_1, x_2)\} \leq p(x_1, x_2) \leq 2 - \theta(x_1, x_2)$.

Remark. The extremal concurrence probability function is very similar to the extremal coefficient function but appears to be more natural and interpretable.

Theorem 4. *Let* $\{\tilde{\eta}(x) \colon x \in \mathcal{X}\}$ *be an independent copy of* $\{\eta(x) \colon x \in \mathcal{X}\}$. *Then for all* $x_1, x_2 \in \mathcal{X}$,

$$p(x_1, x_2) = \mathbb{E}\left[\textit{sign}\{\eta(x_1) - \tilde{\eta}(x_1)\}\textit{sign}\{\eta(x_2) - \tilde{\eta}(x_2)\}\right],$$

i.e., it is the Kendall's τ.

3. Inference

- Suppose we have observed $n = m \times \ell$ independent copies of a stochastic process $\{X(x) \colon x \in \mathcal{X}\}$ at locations $x_1, \ldots, x_k$ and that $X(\cdot)$ belongs to the max-domain of attraction of a max-stable process.
- As usual we partition the data into non overlapping blocks of size m, i.e., the r-th block corresponds to

$$X_{r(m-1)+1}(\cdot), \ldots, X_{r \times m}(\cdot).$$

- And check whether sample concurrence arises in each block, leading to the estimator

$$\hat{p}_m(x_1, \ldots, x_k) = \frac{1}{\ell} \sum_{r=1}^{\ell} 1_{\{\text{sample concurrence in the } r\text{-th block}\}}.$$

- Such a "blocking estimator" usually gives rise to a bias/variance trade-off. As the block size m increases,
  - the bias $p_m - p$ decreases;
  - while the variance $\frac{m}{n} p_m(1 - p_m)$ increases.
- To get the optimal block size m_* we need to compute $p_m - p$ which is usually intractable—apart from specific situations.

Proposition 6. *If* $\{X(x) \colon x \in \mathcal{X}\}$ *is max-stable, then*

$$p_m - p = \sum_{r=2}^{k} \frac{\Pr(|\Theta| = r)}{m^{r-1}}, \qquad m \geq 1.$$

In particular $0 \leq p_m - p \leq (1-p)/m$ *and* $p_m - p \sim m^{-r} c_r$, $r \geq 1$ *and* $c_r > 0$.

- Proposition 6 suggests that the optimal block size satisfies

$$m_* \sim \left\{ \frac{2rc_r^2 n}{p(1-p)} \right\}^{1/(2r+1)}, \qquad \mathrm{MSE}(\hat{p}_{m_*}) \propto n^{-2r/(2r+1)}.$$

Proposition 7. *If* $p \in (0,1)$ *and* $m \sim \lambda n^{1/(2r+1)}$, $\lambda \in (0, \infty)$, *then*

$$\sqrt{\frac{n}{m}}(\hat{p}_m - p) \longrightarrow N\left\{ \frac{c_r}{\lambda^{r+1/2}}, p(1-p) \right\}, \qquad n \to \infty.$$

Remark. When $k = 2$, we can get a unbiased estimator

$$\tilde{p}_m = \frac{m\hat{p}_m - 1}{m - 1},$$

because from Proposition 6 we have $p_m - p = (1-p)/m$.

- $T\{X_1(\mathbf{x}), \dots, X_n(\mathbf{x})\} = \{X_{(1)}(\mathbf{x}), \dots, X_{(n)}(\mathbf{x})\}$ is a sufficient statistic for $p_m(x_1, \dots, x_k)$ where $X_{(1)}(\mathbf{x}) \prec \cdots \prec X_{(n)}(\mathbf{x})$ is the (lexico) sorted sample.

Proposition 8. *Define the new estimator*

$$\hat{p}_m^* = \mathbb{E}(\hat{p}_m \mid T) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \hat{p}_m\{X_{\sigma(1)}, \dots, X_{\sigma(n)}\}.$$

Then $\mathbb{E}(\hat{p}_m^*) = p_m$ *and Var*$(\hat{p}_m^*) \leq$ *Var*$(\hat{p}_m)$.
In addition the estimator can be efficiently computed using

$$\hat{p}_m^* = \frac{1}{\binom{n}{m}} \sum_{i=1}^{n} \binom{d_i}{m-1},$$

where $d_i = \sum_{j=1}^n 1_{\{X_j(\mathbf{x}) \leq X_i(\mathbf{x})\}}$ *and* $\binom{d_i}{m-1} = 0$ *if* $d_i < m - 1$.

Extremal concurrence estimator

- Suppose we have observed n independent copies of a max-stable process $\{\eta(x)\colon x \in \mathcal{X}\}$ at locations $x_1, \dots, x_k$.
- Bivariate extremal concurrence probabilities are estimated using Kendall's τ, i.e.,

$$\hat{p}(x_1, x_2) = \frac{2}{n-1} \sum_{1 \leq i < j \leq n} \text{sign}\{\eta_i(x_1) - \eta_j(x_1)\}\text{sign}\{\eta_i(x_2) - \eta_j(x_2)\},$$

which is an unbiased and asymptotically efficient estimator.

Remark. When $k \geq 3$, we were not able to find any unbiased estimator.

4. Simulation study

Goals

- Shortly we want to investigate
  - how the Rao–Blackwellized estimator $\hat{p}_m^*$ outperforms $\hat{p}_m$—and check the optimal block sizes;
  - the robustness of the extremal concurrence probability estimator $\hat{p}$ to departures from max-stability;
  - our best estimator.

Empirical estimator performances

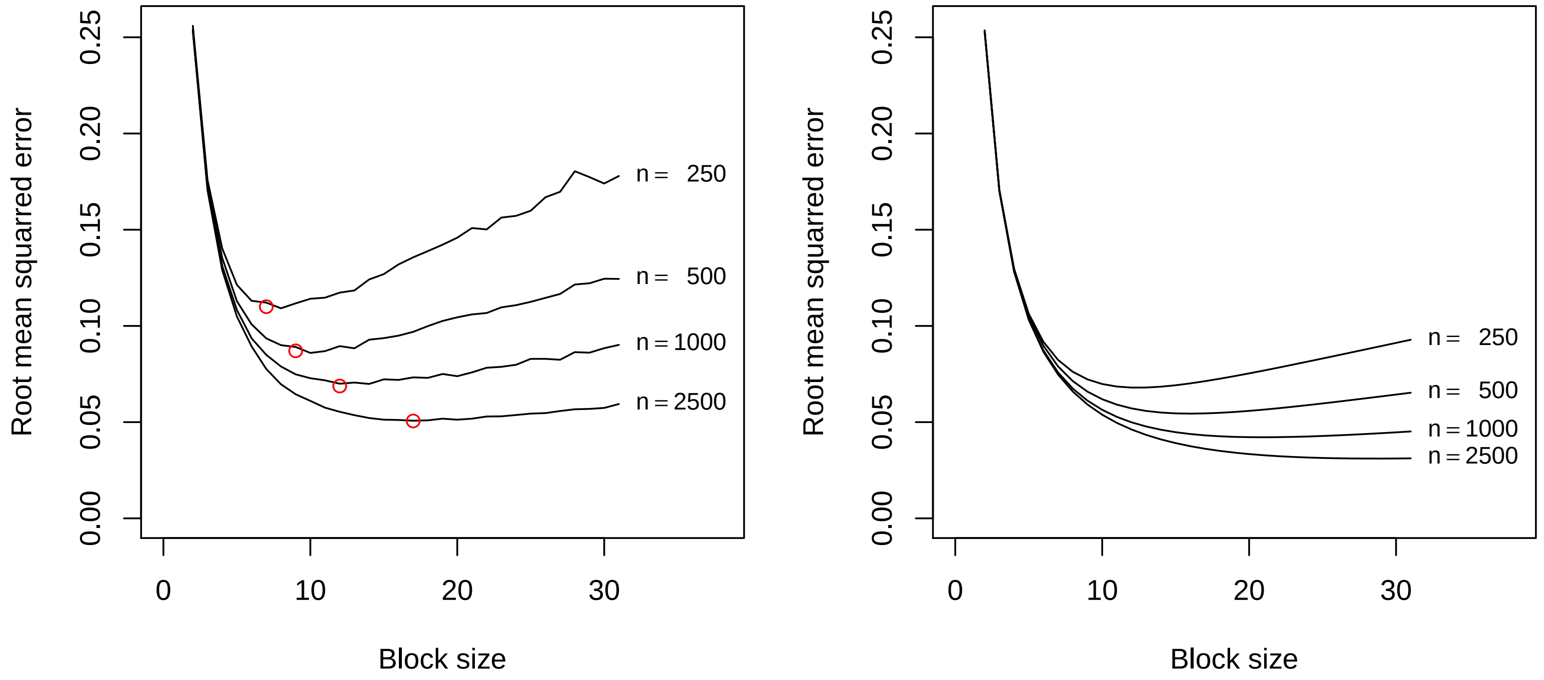

Figure 3: *Evolution of the root mean squared error for $\hat{p}_m$ (left) and $\hat{p}^*_m$ (right) as the block size m and the sample size n increase. These estimates were obtained from 2000 Monte-Carlo samples sampled from a Brown–Resnick model with semivariogram $\gamma(h) = h/1.627$. This semivariogram was chosen such that the theoretical extremal concurrence probability is $p = 0.5$. The red circles indicate the optimal block sizes and their corresponding optimal root mean squared.*

Departures from max-stability

$$\{\tilde{\eta}(x) \colon x \in \mathcal{X}\} = \left\{ \frac{1}{n_0} \max_{i=1,\ldots,n_0} \frac{Y_i(x)}{U_i} \colon x \in \mathcal{X} \right\}, \qquad (1)$$

where $U_1, \ldots, U_{n_0} \overset{\text{iid}}{\sim} U(0,1)$ and $\{Y_i(x) \colon x \in \mathcal{X}, i \geq 1\}$ as in the spectral representation.

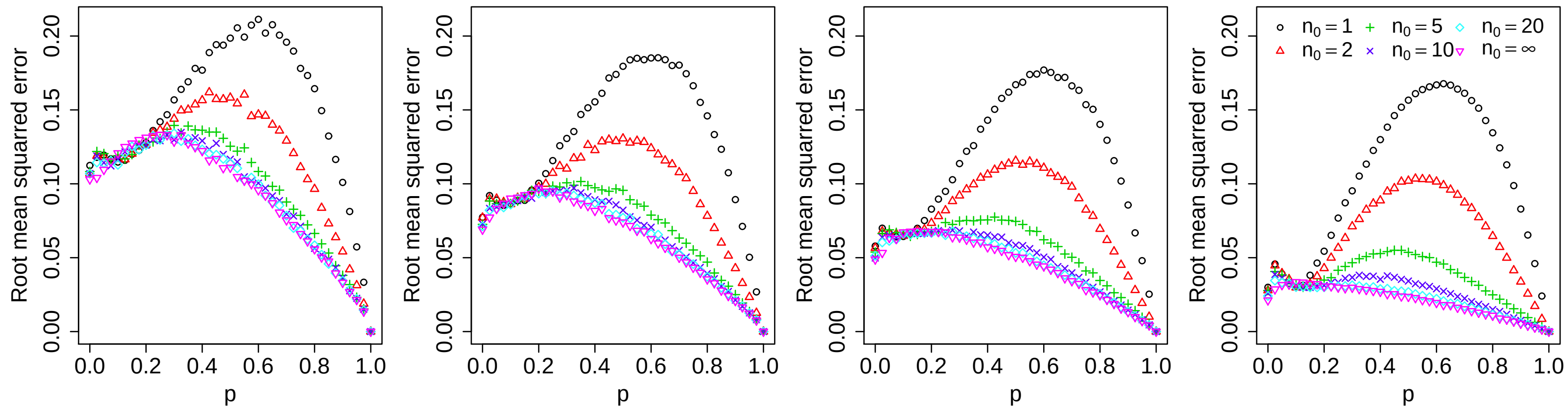

Figure 4: *Evolution of the root mean squared error for $\hat{p}$ as the theoretical extremal concurrence probability p and the number of spectral function n_0 in (1) increase. These estimates were obtained from 2000 Monte-Carlo samples of size n with, from left to right, $n = 25, 50, 100, 500$.*

Overall comparison

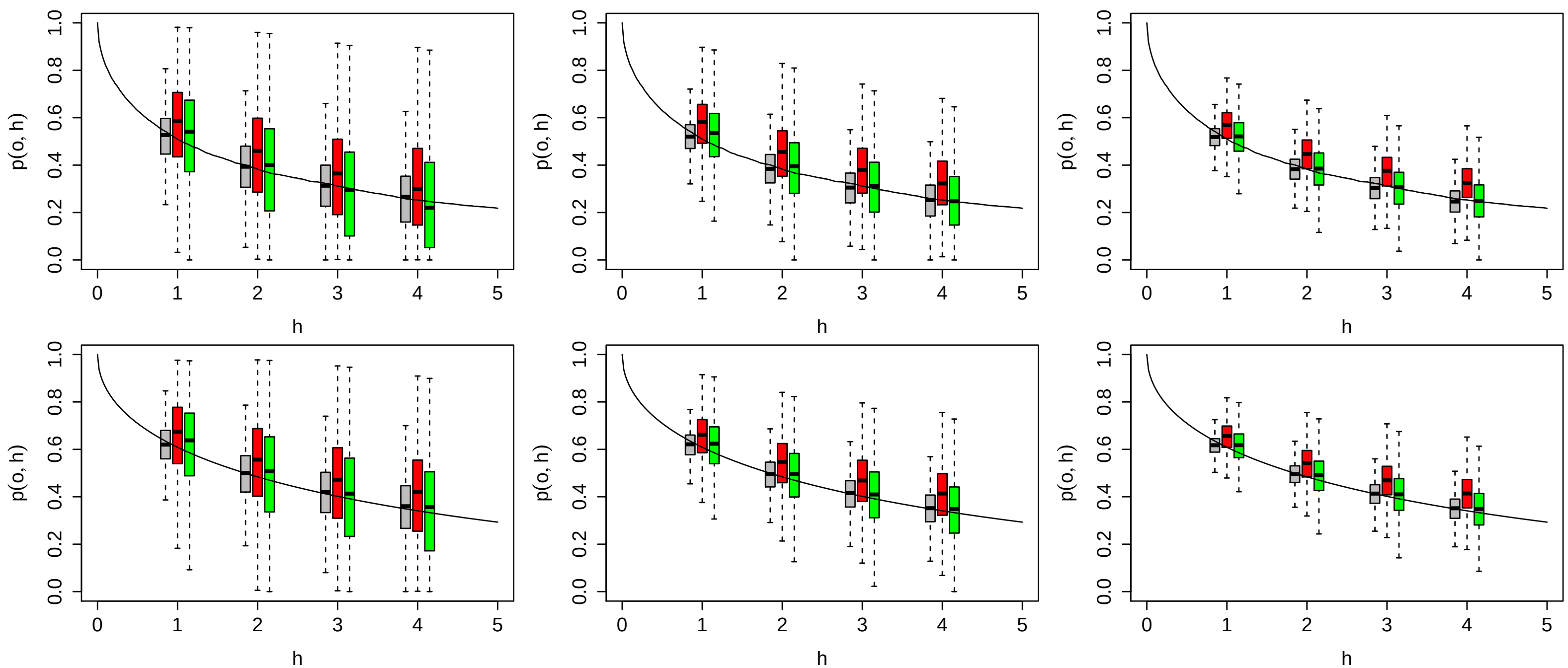

Figure 5: *Boxplots of the sample (red/middle), unbiased sample (green/right) and extremal (grey/left) concurrence probability estimators at distance lags* $h = 1, 2, 3, 4$*. The boxplots were obtained from 2000 independent estimates. From left to right: the sample size is respectively 25, 50, 100 and 500. The top panel corresponds to an extremal-*t *model with* $\nu = 5$*, and correlation function* $\rho(h) = \exp(-h/10)$*. The bottom panel corresponds to a Brown–Resnick model with semi variogram* $\gamma(h) = h/3$*. For each panel, the solid line represents the corresponding theoretical extremal concurrence probability function.*

5. Application

Temperature extremes in continental USA

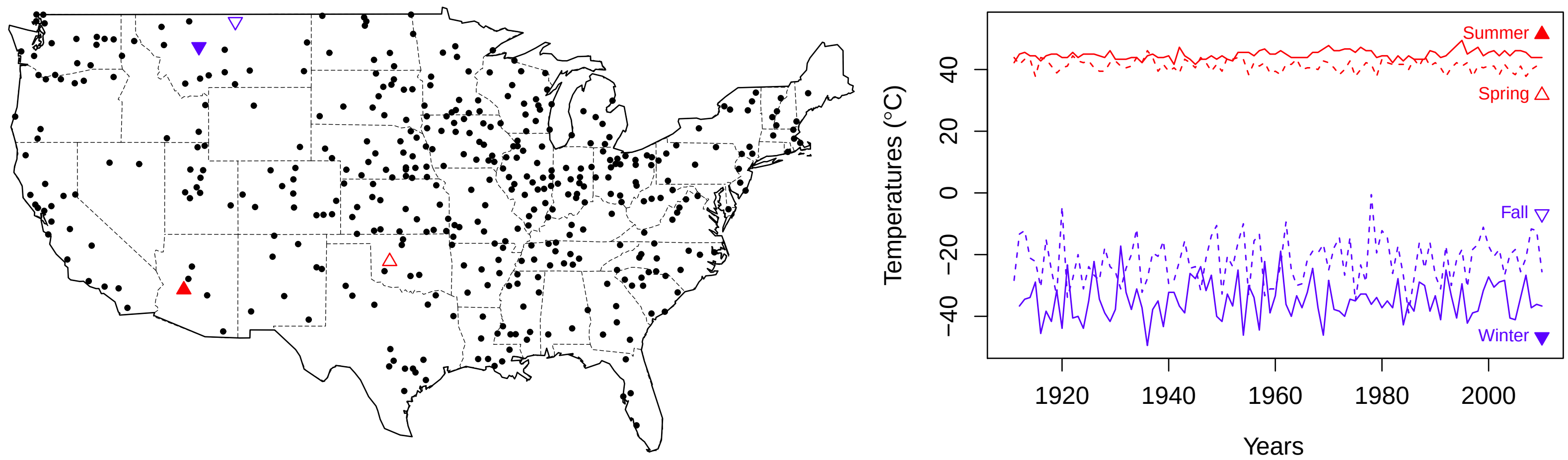

Figure 6: *Left: Spatial distribution of the 424 weather stations. The triangles indicate the selected stations for the analysis—upward: daily maxima, downward: daily minima. Right: The seasonal extrema time series of the selected stations.*

- The data are freely available from http://cdiac.ornl.gov/.
- To avoid any seasonal effect, we will work with seasonal extremes.

Concurrence probability maps

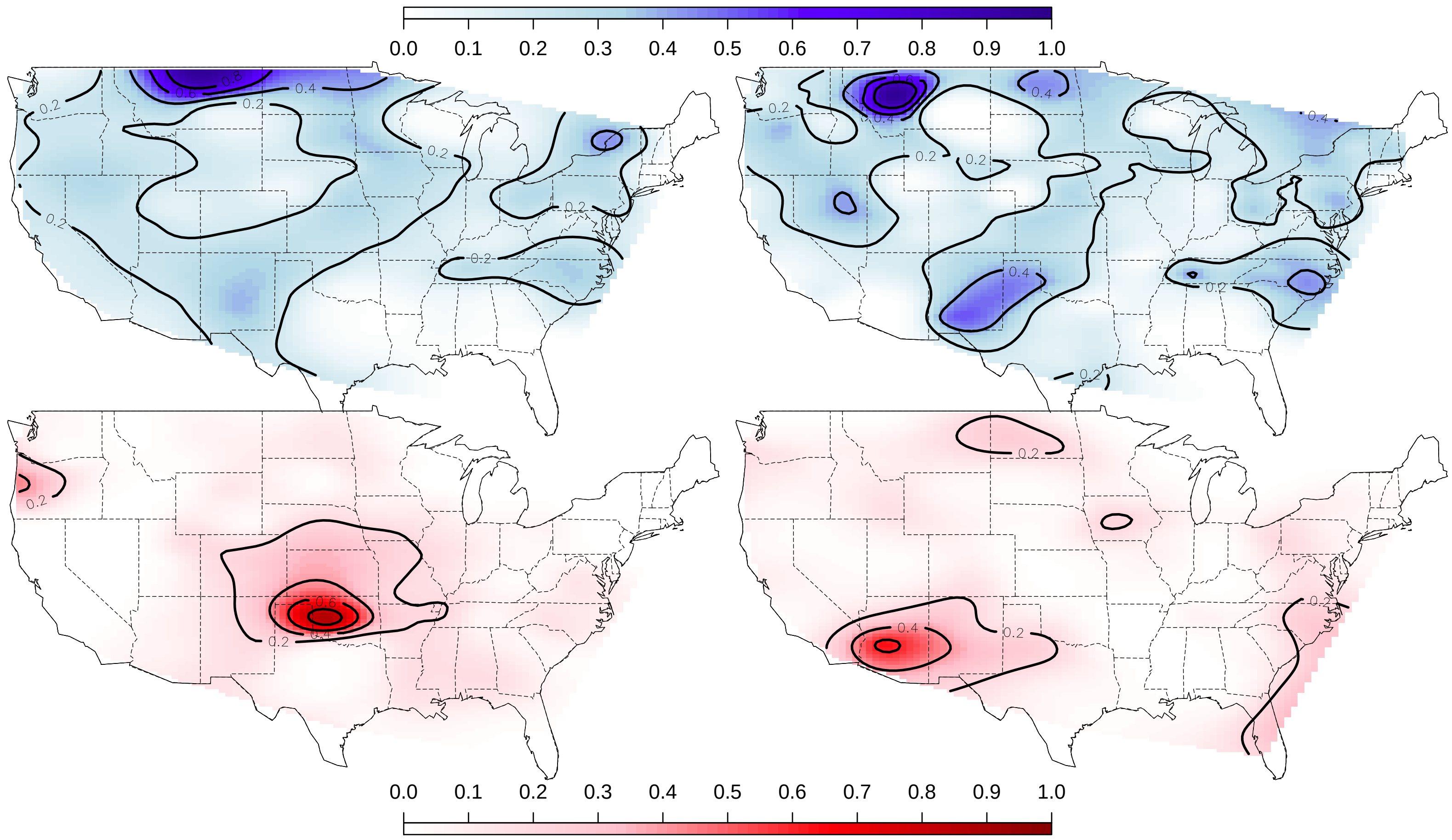

Figure 7: *Maps of the extremal concurrence probability for the four selected stations. Top left: Fall (Sep., Oct. Nov.), top right: Winter (Dec., Jan., Feb.), bottom left: Spring (Mar., Apr., May) and bottom right: Summer (June, July, Aug.).*

Concurrence cells

- For each $x \in \mathcal{X}$ we define the concurrence cell of x as the random set

$$C(x) = \{s \in \mathcal{X} \colon x \text{ and } s \text{ are concurrent}\}.$$

- Clearly we have

$$\mathbb{E}\{|C(x)|\} = \mathbb{E}\left[\int_{\mathcal{X}} 1_{\{s \in C(x)\}} \mathrm{d}s\right] = \int_{\mathcal{X}} p(x, s) \mathrm{d}s.$$

- This suggests plotting the spatial distribution of the pointwise expected concurrence cell area, i.e.,

$$\{(s, \mathbb{E}\{|C(s)|\}) \colon s \in \mathcal{X}\}.$$

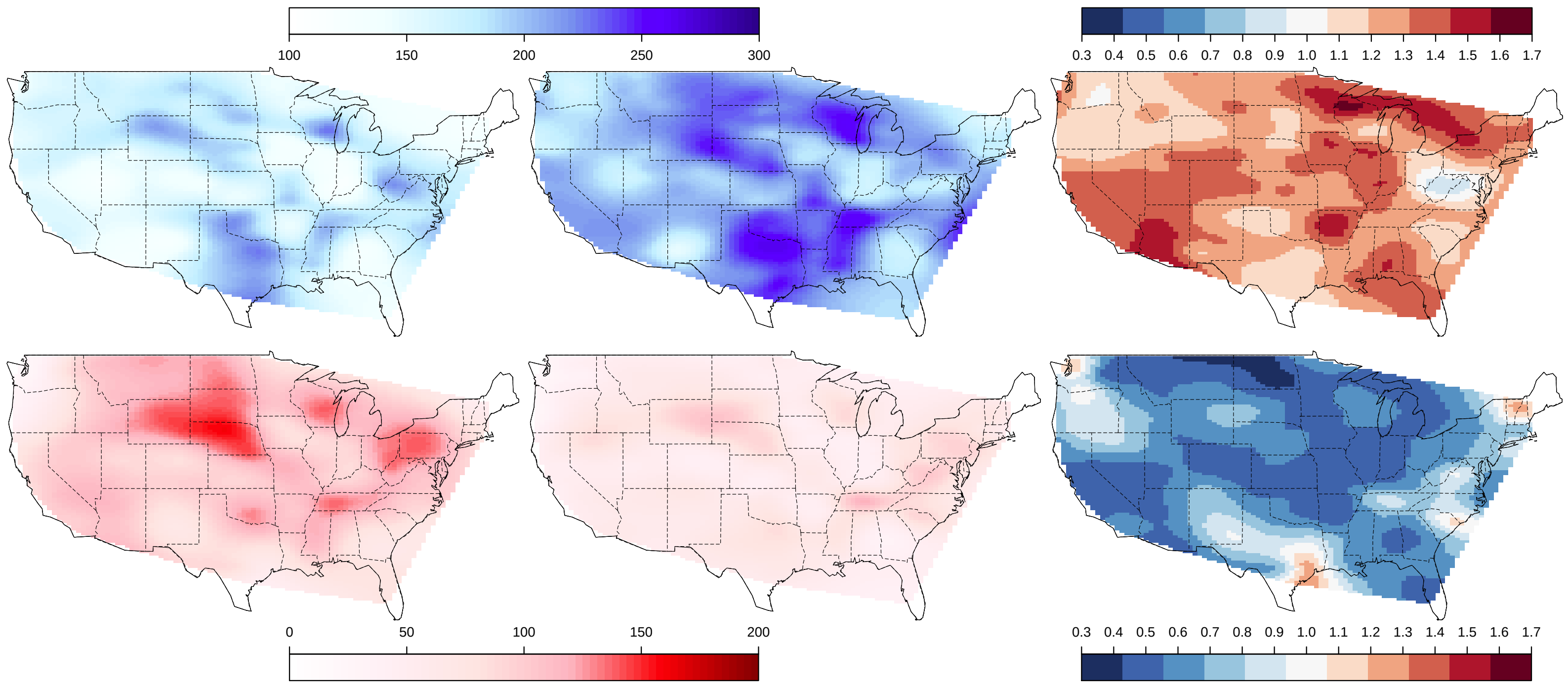

Figure 8: *Estimated spatial distribution of the expected extremal concurrence cell areas—in squared degree, i.e., around* 1000 *km*^{2}*. From left to right: 1910–1950, 1951–2010, and their ratio (1951–2010 at the numerator). Top: Winter minima, bottom: Summer maxima.*

El Niño/La Niña impact

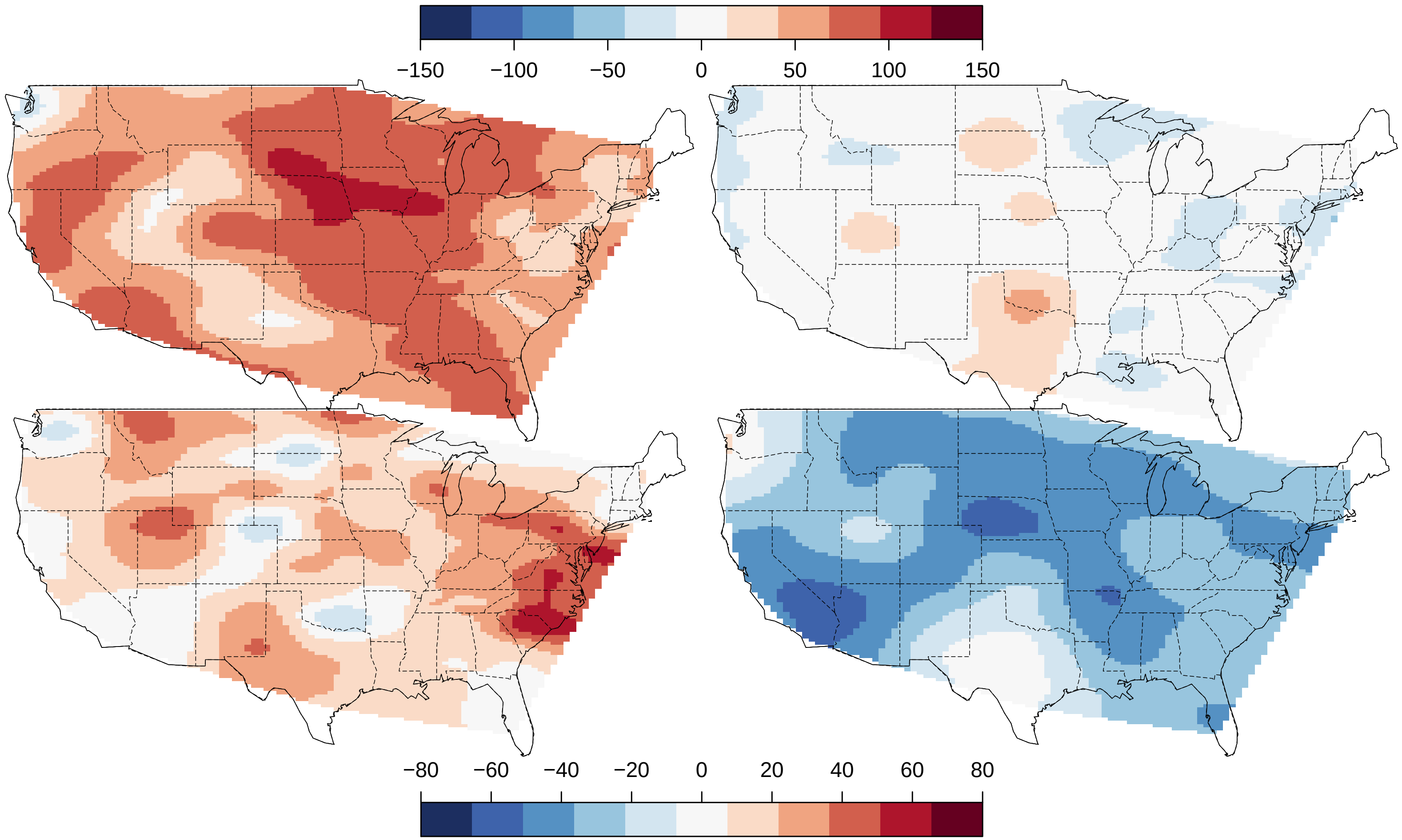

Figure 9: *Spatial distribution of the estimated concurrence cell areas anomalies in squared degree, i.e., around* 1000 *km*^{2} *for winter minima (top) and summer maxima (bottom). The data were stratified into three classes: El Niño, La Niña and the base class "La Nada". Left panels: El Niño anomalies. Right panels: La Niña anomalies.*

THANK YOU FOR YOUR ATTENTION!

Dombry, C., Ribatet, M. and Stoev, S. (2015) Probabilities of concurrent extremes. Submitted. http://arxiv.org/abs/1503.05748

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