Statistical Modelling of Spatial Extremes

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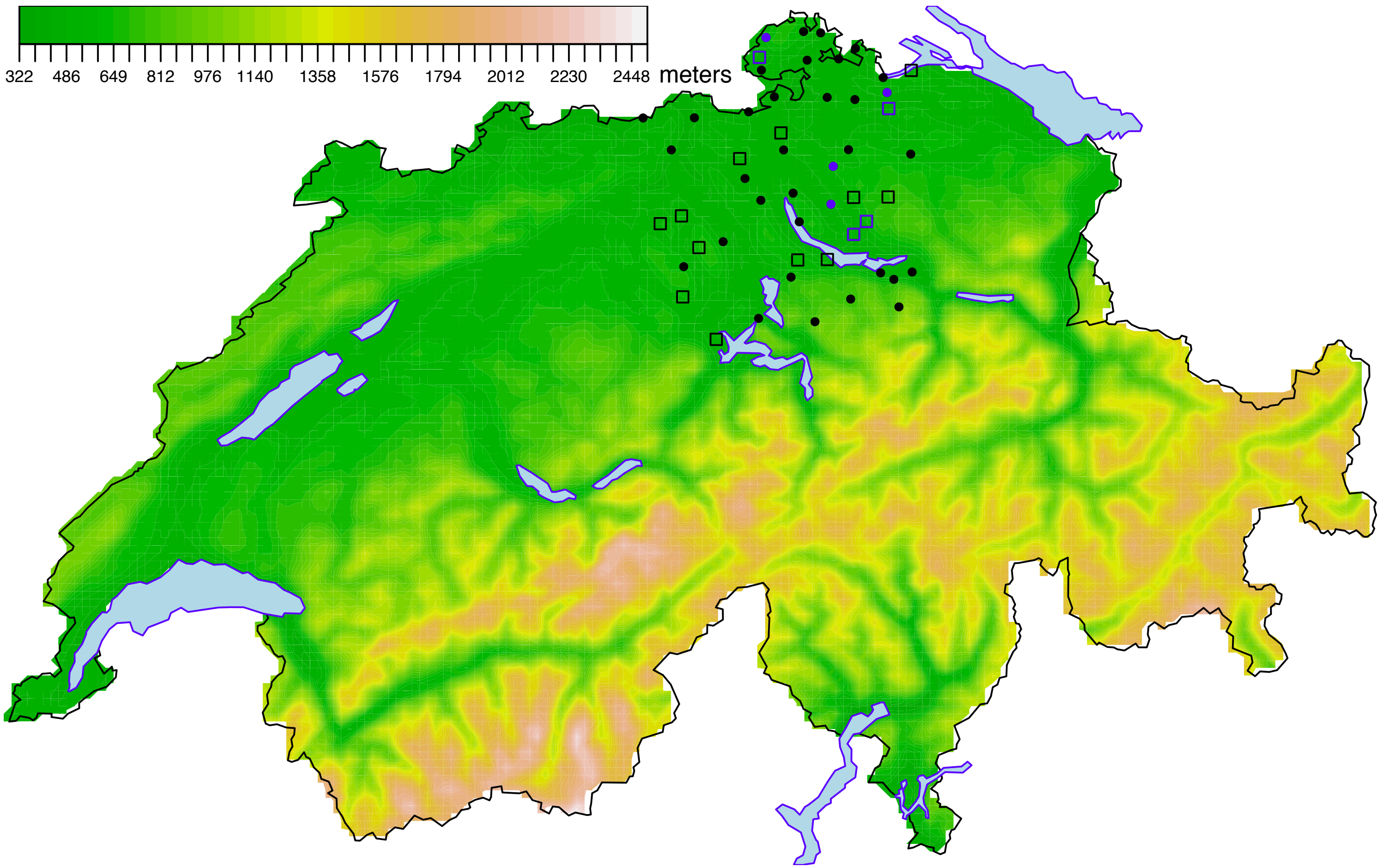

Figure 1: *Map of Switzerland showing the stations of the 51 rainfall gauges used for the analysis, with an insert showing the altitude. The 36 stations marked by circles were used to fit the models, and those marked with squares were used to validate the models. The pairs of stations with blue symbols will appear in the next Figure.*

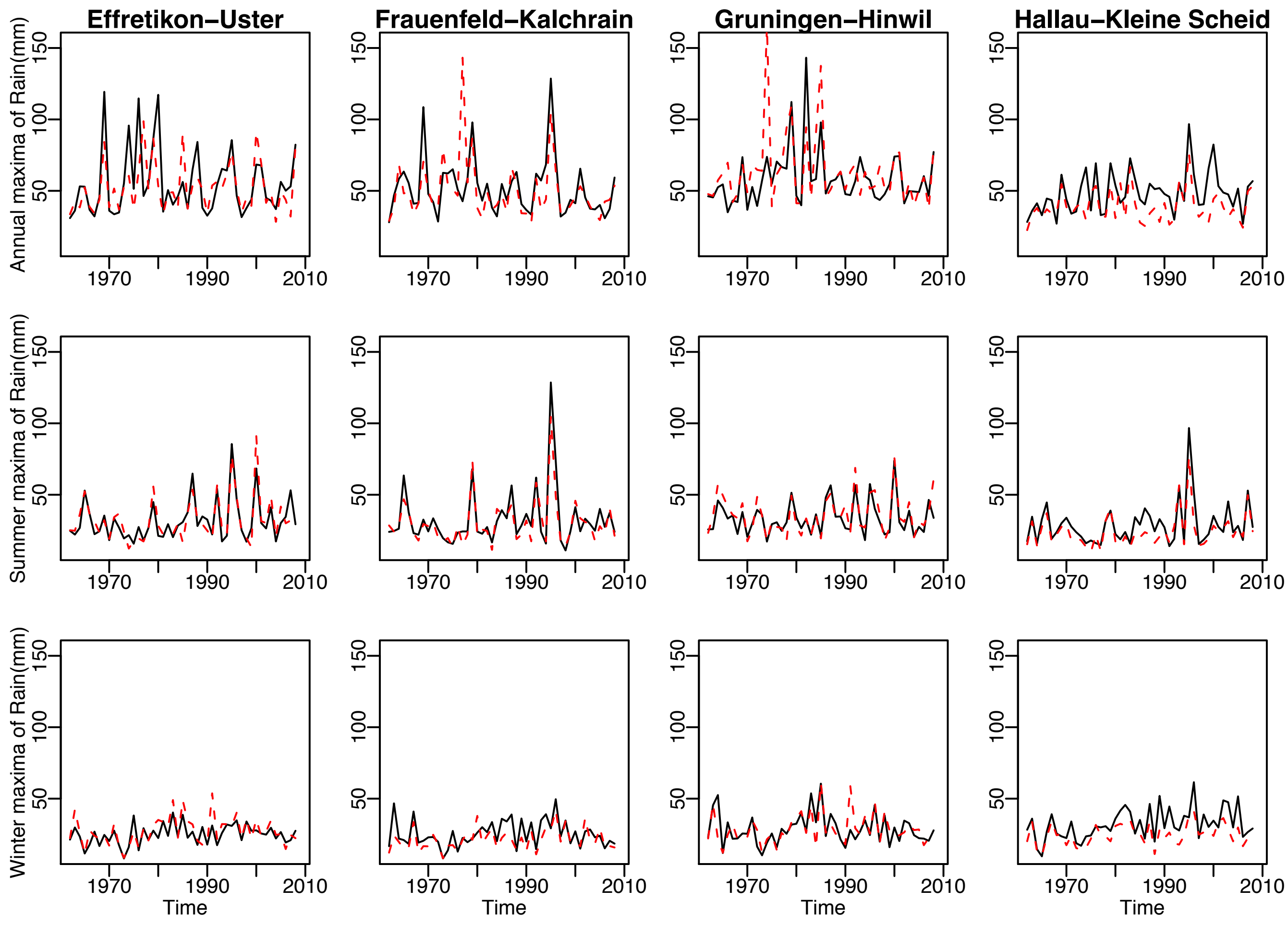

Figure 1: *Annual, summer and winter maximum daily rainfall values for 1962–2008 at the four pairs of stations shown in blue in the previous Figure.*

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EVT: Finite dimensional setting

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- Let $X_1, X_2, \ldots$ be independent replications from F
- Provided that G is non degenerate

$$\Pr\left[\max_{i=1,\ldots,n} \frac{X_i - b_n}{a_n} \leq x\right] \longrightarrow G(x), \qquad n \to +\infty, \qquad (1)$$

for some normalizing sequences $a_n > 0$ and $b_n \in \mathbb{R}$, then

$$G(x) = \exp\left\{-(1+\xi x)^{-1/\xi}\right\}.$$

- For modelling purposes, as long as n is large enough we will assume

$$\Pr\left[\max_{i=1,\ldots,n} X_i \leq x\right] \approx \exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}\right\}.$$

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- Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ iid d-random vectors with distribution F
- Our interest is in the (non degenerate) limiting distribution

$$\Pr\left[\max_{i=1,\dots,n} \frac{\mathbf{X}_i - \mathbf{b}_n}{\mathbf{a}_n} \leq \mathbf{x}\right] \longrightarrow G(\mathbf{x}), \qquad n \to \infty$$

for some sequences $\mathbf{a}_n > \mathbf{0}$ and $\mathbf{b}_n \in \mathbb{R}^d$. G is called a multivariate extreme value distribution

- Paralleling the univariate case we have

$$G^t(\mathbf{x}) = G\{\boldsymbol{\alpha}(t)\mathbf{x} + \boldsymbol{\beta}(t)\}, \qquad t > 0,$$

for some normalizing functions $\boldsymbol{\alpha}(t) > \mathbf{0}$ and $\boldsymbol{\beta}(t) \in \mathbb{R}^d$.

- W.l.o.g. we'll assume unit Fréchet margins, i.e.,

$$G(x, +\infty, \dots, +\infty) = \dots = G(+\infty, \dots, +\infty, x) = \exp(-1/x),$$

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Theorem. *Let* $E = [0, +\infty]^d \setminus \{\mathbf{0}\}$*.* G *is a unit Fréchet MEVD iff there exists a finite measure* H *on* $\mathbb{S}_d = \{\mathbf{y} \in E : \|\mathbf{y}\| = 1\}$ *such that*

$$\int_{\mathbb{S}_d} \omega_i dH(\boldsymbol{\omega}) = 1, \qquad G(\mathbf{x}) = \exp\left\{ - \int_{\mathbb{S}_d} \max_{i=1,\dots,d} \frac{\omega_i}{x_i} dH(\boldsymbol{\omega}) \right\},$$

for $i = 1, \dots, d$ *and* $\mathbf{x} \in E$*.*
Equivalently $G(\mathbf{x}) = \exp\{-V(x_1, \dots, x_d)\}$ *where* V *is homogeneous of order* -1*, i.e.* $V(t\,\cdot) = t^{-1}V(\cdot)$*, and* $V(x, +\infty, \dots, +\infty) = \dots = V(+\infty, \dots, +\infty, x) = x^{-1}$*.*

Remark. Let $\mathbf{x} = (x, \dots, x)$, $x > 0$. As V is homogeneous,

$$G(\mathbf{x}) = \exp\{-V(x, \dots, x)\} = \exp(-\theta_d / x) = G(x)^{\theta_d},$$

where $\theta_d = V(1, \dots, 1)$ is known as the extremal coefficient.

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Theorem. *Let $E = [0, +\infty]^d \setminus \{\mathbf{0}\}$. G is a unit Fréchet MEVD iff there exists a finite measure H on $\mathbb{S}_d = \{\mathbf{y} \in E : \|\mathbf{y}\| = 1\}$ such that*

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for $i = 1, \dots, d$ and $\mathbf{x} \in E$.
Equivalently $G(\mathbf{x}) = \exp\{-V(x_1, \dots, x_d)\}$ where V is homogeneous of order -1, i.e. $V(t\,\cdot) = t^{-1}V(\cdot)$, and $V(x, +\infty, \dots, +\infty) = \dots = V(+\infty, \dots, +\infty, x) = x^{-1}$.

Remark. In a spatial context, it is more convenient to think of the extremal coefficient as a function of the distance between to points in $\mathbb{R}^d$. This is the extremal coefficient function

$$\theta(h) = -z \log \Pr[Z(o) \le z, Z(h) \le z], \qquad z > 0.$$

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Moving smoothly to the infinite dimensional case

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- In trying to model spatial extremes, we aim at capturing the
  1. spatial behavior of the marginal parameters, i.e., μ, σ, ξ
  2. spatial dependence, e.g., a single storm impacts several locations
- For the first point, one might use polynomial surfaces, e.g.,

$$\mu(x) = \beta_0 + \beta_1 \text{lon}(x) + \beta_2 \text{lat}(x)$$

- For the second point, there are several possibilities based on the model used

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- Deterministic trend surfaces might not be flexible enough to capture the spatial variability of the marginal parameters
- What if we use instead stochastic processes for this? E.g.,

$$\mu(x) = f_\mu(x; \boldsymbol{\beta}_\mu) + S_\mu(x; \alpha_\mu, \lambda_\mu),$$

  where f_μ is a deterministic function and S_μ is a zero mean Gaussian process.
- Then conditional on the values of the 3 Gaussian processes at the sites $(x_1, \ldots, x_K)$,

$$Y_i(x_j) \mid \{\mu(x_j), \sigma(x_j), \xi(x_j)\} \sim \text{GEV}\{\mu(x_j), \sigma(x_j), \xi(x_j)\},$$

  independently for each location $(x_1, \ldots, x_K)$.
- This is most naturally performed in a MCMC framework

Assets and Drawbacks: BHM

- The quantile surfaces are realistic BUT
- After averaging over $S(x)$ the marginal distribution of $\{Y(x)\}$ isn't GEV
- The spatial dependence is ignored because of conditional independence

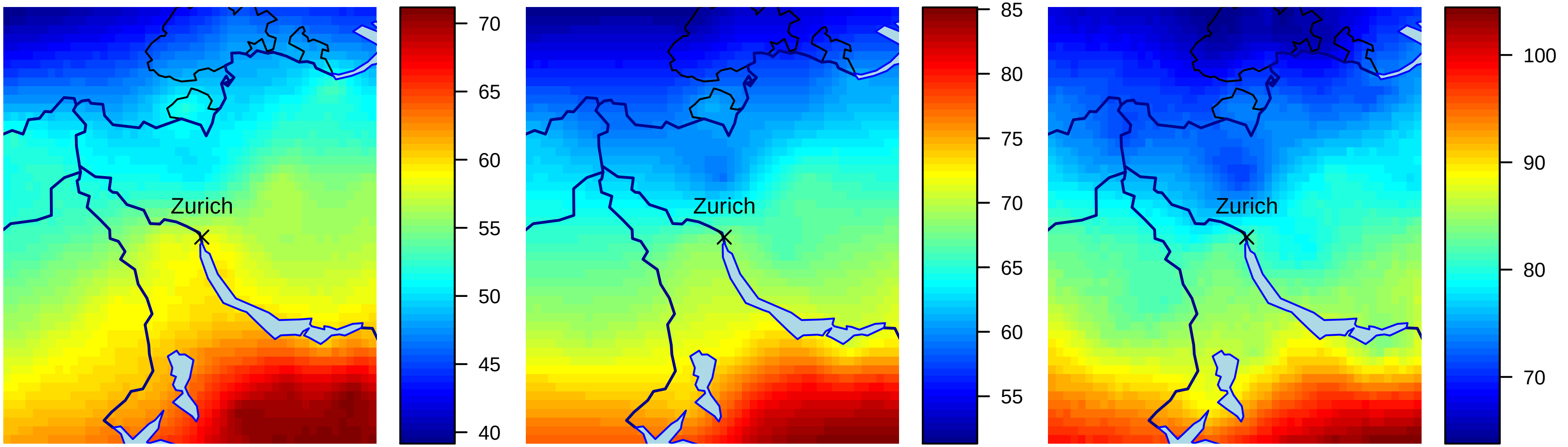

Figure 2: *Maps of the pointwise 25-year return levels for rainfall (mm) obtained from the latent variable model. The left and right panels are respectively the estimated* 0.025 *and* 0.975 *quantiles, and the middle panel shows the posterior mean.*

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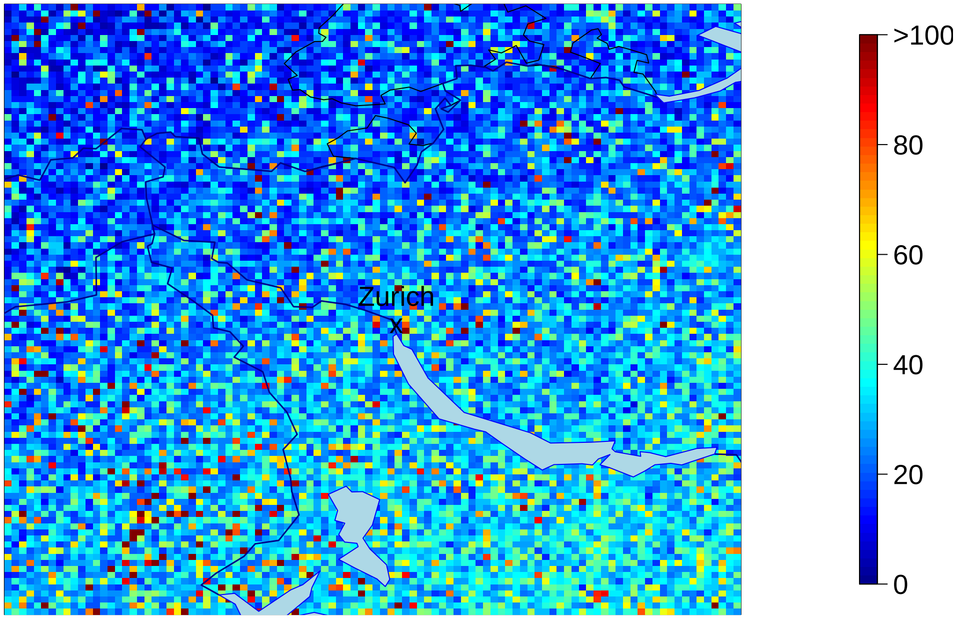

Figure 2: *One realisation of the latent variable model, showing the lack of local spatial structure.*

Assets and Drawbacks: BHM

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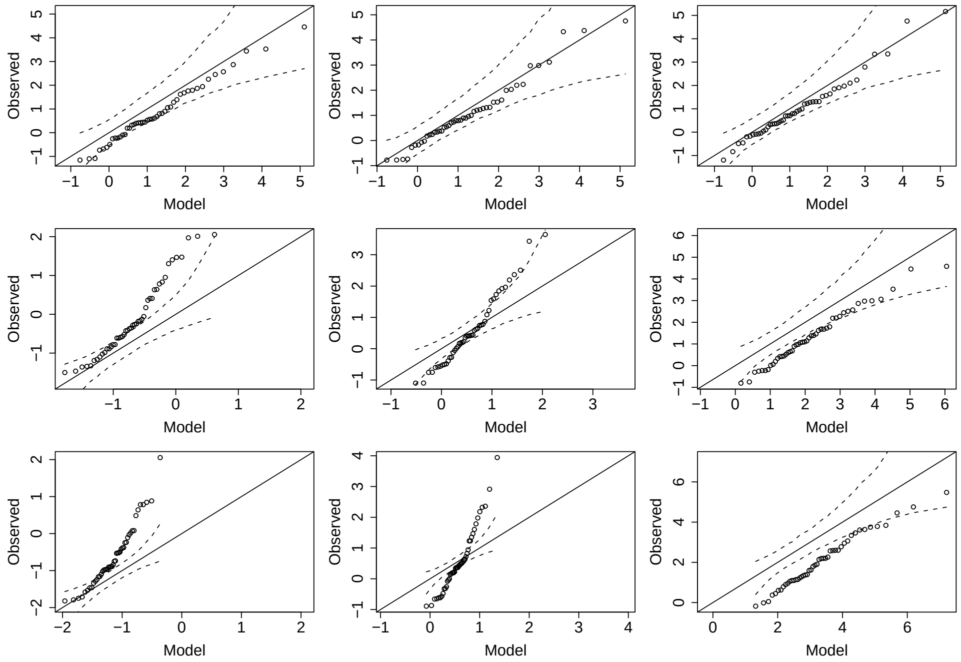

Figure 2: *Model checking for the Bayesian hierarchical model.*

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- One might be tempted to use copula to take into account the spatial dependence
- For instance using a Gaussian copula

$$\Pr[Y(x_1) \leq y_1, \ldots, Y(x_K) \leq y_K] = \Phi\left\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_K)\right\}$$

or a t-copula

$$\Pr[Y(x_1) \leq y_1, \ldots, Y(x_K) \leq y_K] = T_\nu\left\{T_\nu^{-1}(u_1), \ldots, T_\nu^{-1}(u_K)\right\}$$

where $u_i = \text{GEV}\{y_i; \mu(x_i), \sigma(x_i), \xi(x_i)\}$ for all $i = 1, \ldots, K$.

Assets and Drawbacks: Copula

- On the "copula scale": the dependence seems more or less OK
- But this is no longer true at the original, i.e., extremal, scale.

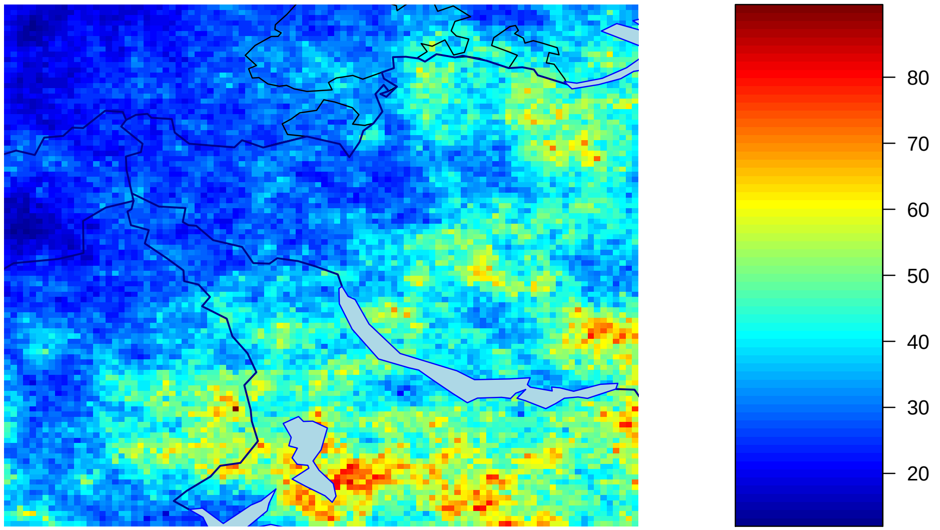

Figure 3: *One simulation from the fitted Gaussian copula model.*

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- On the "copula scale": the dependence seems more or less OK
- But this is no longer true at the original, i.e., extremal, scale.

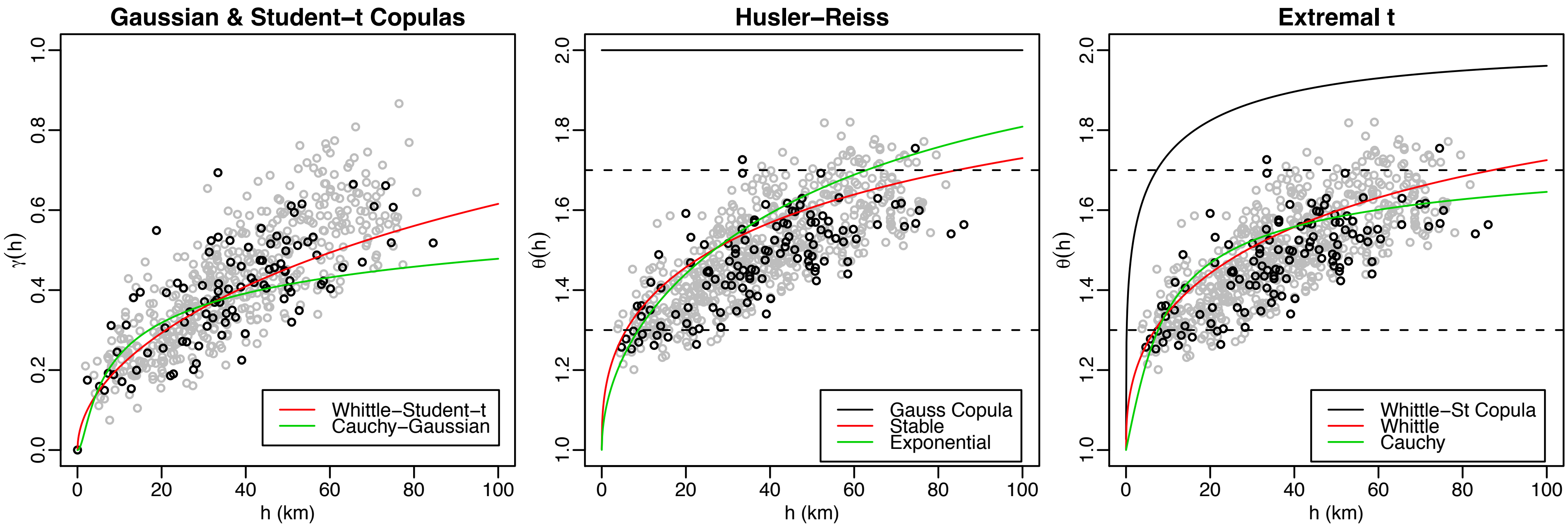

Figure 3: *Comparison between the empirical variogram and the fitted one (red line) on the Gaussian/Student scale. On the original scale we compare the extremal coefficient function. Grey points: data used for model fitting, black ones: data used for model validation.*

- On the “copula scale”: the dependence seems more or less OK
- But this is no longer true at the original, i.e., extremal, scale.

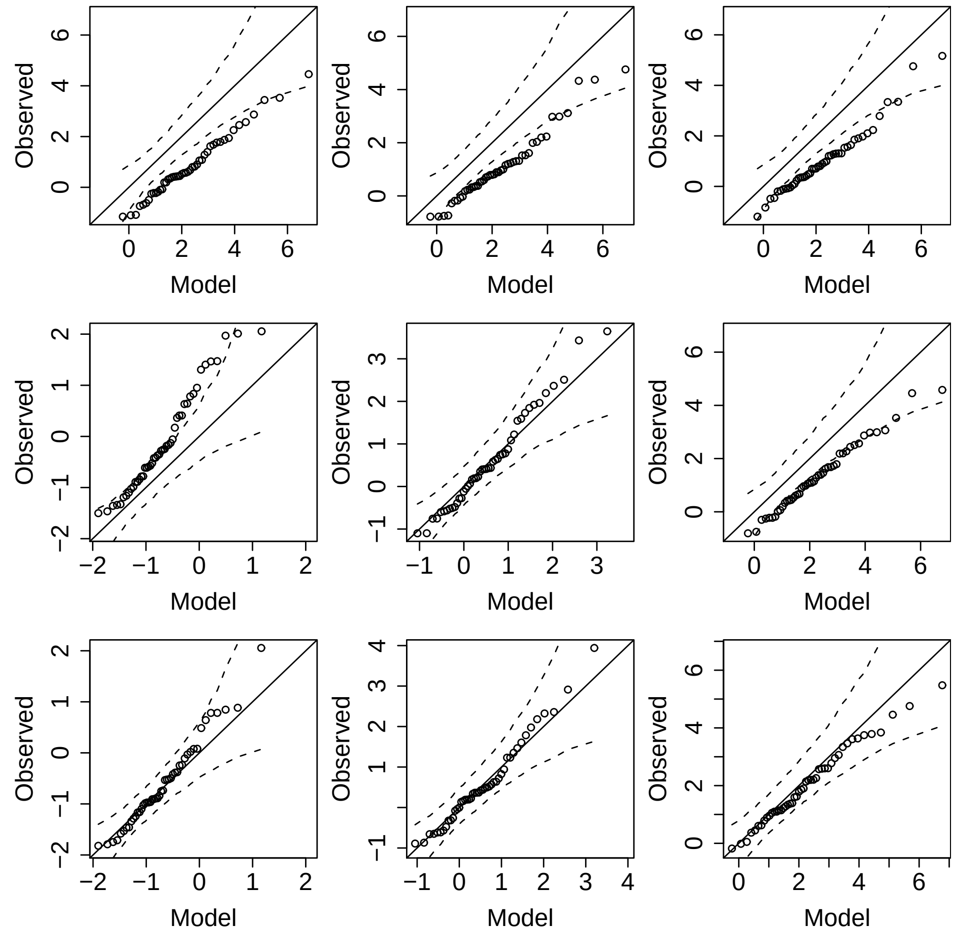

Figure 3: *Model checking for the gaussian copula model.*

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- These two models all fail to capture some aspect of the data

Latent	More realistic quantile surfaces but still no spatial dependence modelling
Copula	Might falsely take into account the spatial dependence — if not max-stable!

- How one can model spatial extremes?

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Why max-stable processes?

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- Let $\{Y(x)\}_{x \in K}$ be a continuous sample path stochastic process and $Y_1, \ldots, Y_n$ independent replicates of it
- Our goal is to focus on the (non degenerate) limiting process

$$\left\{ \max_{i=1,\ldots,n} \frac{Y_i(x) - b_n(x)}{a_n(x)} \right\}_{x \in \mathbb{R}^d} \xrightarrow{\text{d}} \{Z(x)\}_{x \in \mathbb{R}^d}, \qquad n \to +\infty,$$

where $a_n(x) > 0$ and $b_n(x)$ are sequences of continuous functions.

de Haan [1984] shows that the class of the limiting process $\{Z(x)\}_{x \in K}$ corresponds to that of max-stable processes.

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Definition. A stochastic process $\{Z(x)\}_{x \in K}$ with continuous sample paths is called max-stable if there are continuous functions $a_n(x) > 0$ and $b_n(x) \in \mathbb{R}$ such that if $Z_1, \ldots, Z_n \overset{iid}{\sim} Z$ then

$$\max_{i=1,\ldots,n} \frac{Z_i(\cdot) - b_n(\cdot)}{a_n(\cdot)} \overset{\mathrm{d}}{=} Z(\cdot), \qquad i = 1, 2, \ldots$$

Remark. If $\{Z(x)\}_{x \in K}$ has unit Fréchet margins then the above equation becomes

$$n^{-1} \max_{i=1,\ldots,n} Z_i(\cdot) \overset{\mathrm{d}}{=} Z(\cdot), \qquad i = 1, 2, \ldots$$

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- Probably the most useful spectral representation of max-stable processes is the following.

Theorem (Schlather, 2002). *Let* $\{\xi_i\}_{i\geq 1}$ *be the points of a Poisson process on* $(0, +\infty]$ *with intensity* $d\Lambda(\xi) = \xi^{-2}d\xi$ *and* $Y_1, Y_2, \ldots$ *be i.i.d. replications of a stochastic process such that* $\mathbb{E}[\max\{0, Y_i(x)\}] = 1$, *for all* $x \in \mathbb{R}^d$. *The processes* Y_i *and the points of the Poisson process are assumed to be independent. Then*

$$Z(\cdot) = \max_{i\geq 1} \xi_i \max\{0, Y_i(\cdot)\}.$$

is a max-stable process with unit Fréchet margins.

- Suitable choices for $Y(\cdot)$ yield different max-stable processes.

Some models

Smith $Y_i(x) = \varphi(x - U_i)$, $\{U_i\}_{i \geq 1}$ points of a homogeneous PP on $\mathbb{R}^d$
Schlather $Y_i(x) = \sqrt{2\pi}\varepsilon_i(x)$, $\varepsilon_i(\cdot)$ standard Gaussian process
Geometric $Y_i(x) = \exp\{\sigma\varepsilon_i(x) - \sigma^2/2\}$
Brown–Res. $Y_i(x) = \exp\{\epsilon_i(x) - \gamma(x)\}$, $\epsilon_i(\cdot)$ intrinsically stationary Gaussian process with (semi) variogram γ.

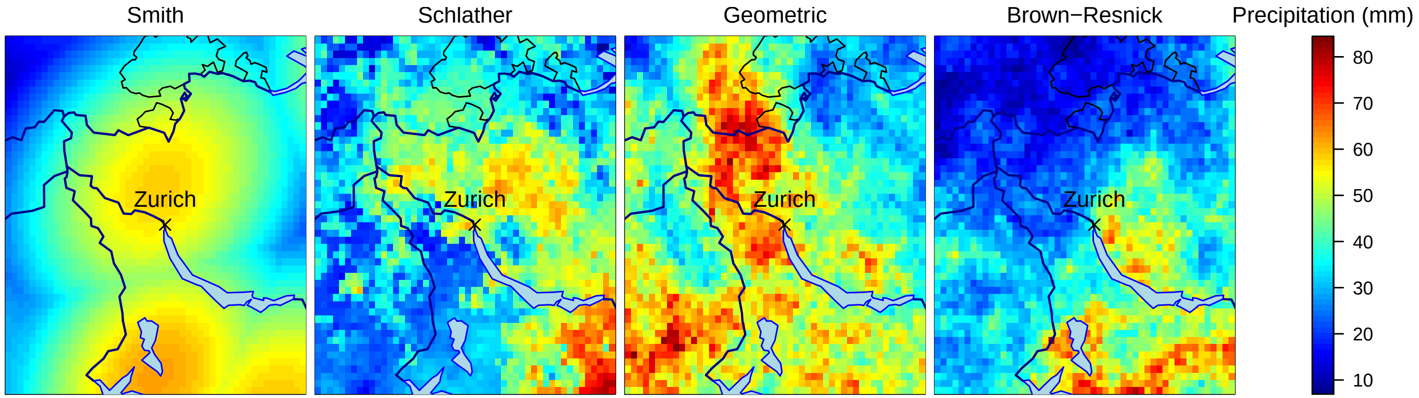

Figure 4: *One realization of a max-stable process. From left to right: extremal coefficient functions, Smith's, Schlather's, Geometric and Brown–Resnick's models.*

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- It would be nice to have a kind of variogram for extremes of a stochastic process

$$\gamma(x_1 - x_2) = \frac{1}{2}\mathbb{E}[\{Z(x_1) - Z(x_2)\}^2]$$

- But if we assume that Z is a unit Fréchet max-stable process, Var$[Z(x)] = \mathbb{E}[Z(x)] = +\infty$!

Theorem (Cooley et al., 2006). *If $\{Z(x)\}_{x \in K}$ is a unit Fréchet max-stable process, then*

$$2\nu_F(x_1 - x_2) := \mathbb{E}\left[\left|F\left(Z(x_1)\right) - F\left(Z(x_2)\right)\right|\right] = \frac{\theta(x_1 - x_2) - 1}{\theta(x_1 - x_2) + 1}.$$

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Smith $\theta(h) = 2\Phi\left(\frac{\sqrt{h^T \Sigma^1 h}}{2}\right)$

Schlather $\theta(h) = 1 + \sqrt{\frac{1-\rho(h)}{2}}$

Geometric $\theta(h) = 2\Phi\left(\sqrt{\frac{\sigma^2\{1-\rho(h)\}}{2}}\right)$

Brown–Resnick $\theta(h) = 2\Phi\left(\sqrt{\frac{\gamma(h)}{2}}\right)$

- Constraints on positive definite function [Matérn, 1986] implies that Schlather has $\theta(h) \leq 1.838$ and that $\theta(h) \longrightarrow 1 + \sqrt{1/2}$ as $h \to +\infty$: independence never reached!
- Similarly for the Geometric model, $\theta(h) \leq 2\Phi(0.838\sigma)$ and $\theta(h) \longrightarrow 2\Phi(\sigma/\sqrt{2})$: independence might be virtually reached if σ^2 is large enough.
- If γ is unbounded, independence is always reached.

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- Recall that the spectral representation is

$$Z(x) = \max_{i \geq 1} \xi_i \max\{0, Y_i(x)\},$$

  where $\{\xi_i\}_{i\geq 1}$ are the points of a Poisson process on $(0, +\infty]$ with intensity $\mathrm{d}\Lambda(\xi) = \xi^{-2}\mathrm{d}\xi$ and $Y_i \stackrel{iid}{\sim} Y$ where Y satisfies $\mathbb{E}[\max\{0, Y(x)\}] = 1$.
- Simulation of an infinite number of points of a Poisson process and independent replications of Y are required — ouch!
- Under further assumptions on Y it is however possible to get exact simulations.

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1. Start with a standard PP on $(0, +\infty]$, i.e., $\sum E_i$, $E_i \overset{iid}{\sim}$ Exp(1) — intensity $\tilde{\Lambda}([a, b]) = b - a$
2. Apply the mapping $T : x \mapsto x^{-1}$ to the above points, this gives a new PP with intensity measure

$$\Lambda([a, b]) = \tilde{\Lambda}\{T^{-1}([a, b])\} = \tilde{\Lambda}([b^{-1}, a^{-1}]) = a^{-1} - b^{-1}$$

   and its intensity density is as required $\mathrm{d}\Lambda(\xi) = \xi^{-2}\mathrm{d}\xi$.
3. But $\xi_n \overset{d}{=} 1/\sum_{i=1}^{n} E_i \downarrow 0^+$ as $n \to +\infty$, so if Y is uniformly bounded by $C < +\infty$ then

$$0 \leq \xi_i Y(x) \leq \xi_i C \downarrow 0^+, \qquad n \to +\infty$$

4. And we only need a finite number of replications

□ When Y isn't stationary like for Brown–Resnick processes

$$Y(x) = \exp\{\varepsilon(x) - \gamma(x)\}, \qquad \gamma(h) \propto h^{\alpha},\, 0 < \alpha \leq 2,$$

the above algorithm give poor approximations

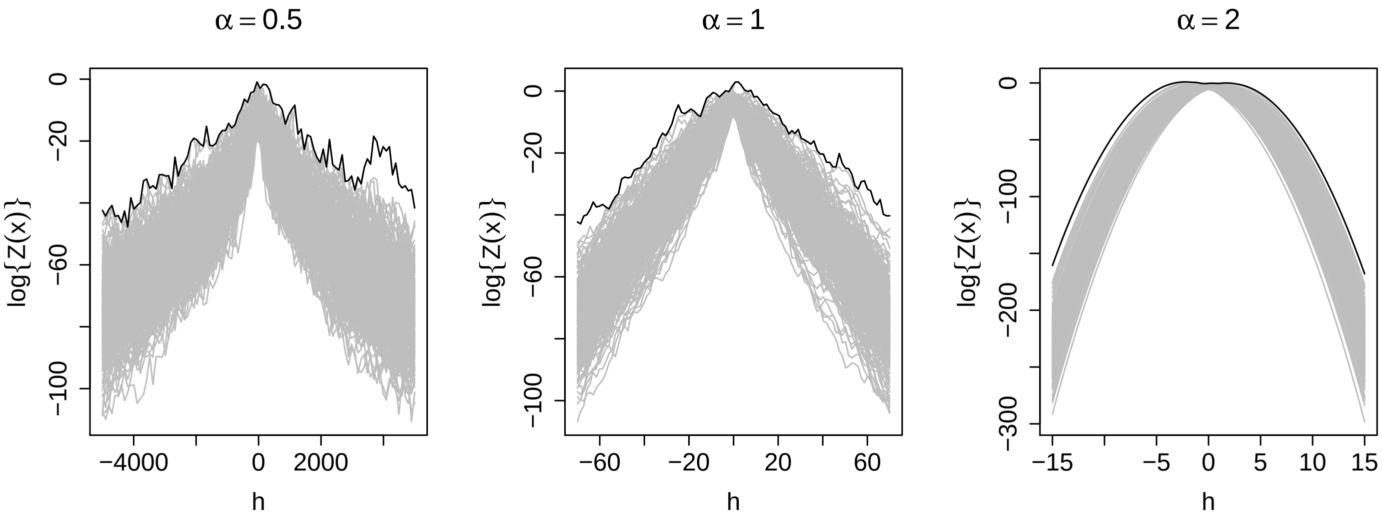

Figure 5: *Simulation of Brown–Resnick processes when ε is a fraction Brownian motion. These simulation are based on $m = 250$ independent simulations (grey curves). The black curves corresponds to the simulated Brown–Resnick processes.*

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- Since B.–R. processes are stationary one can use

$$Z(x) = \max_{i \geq 1} \xi_i \exp\{\varepsilon_i(x - U_i) - \gamma(x - U_i)\}, \qquad U_i \stackrel{iid}{\sim} F \text{ arbitrary.}$$

- Roughly speaking the random shifting $x \mapsto x - U$ mitigates the impact of the conditioning $\varepsilon(o) = 0$ a.s.

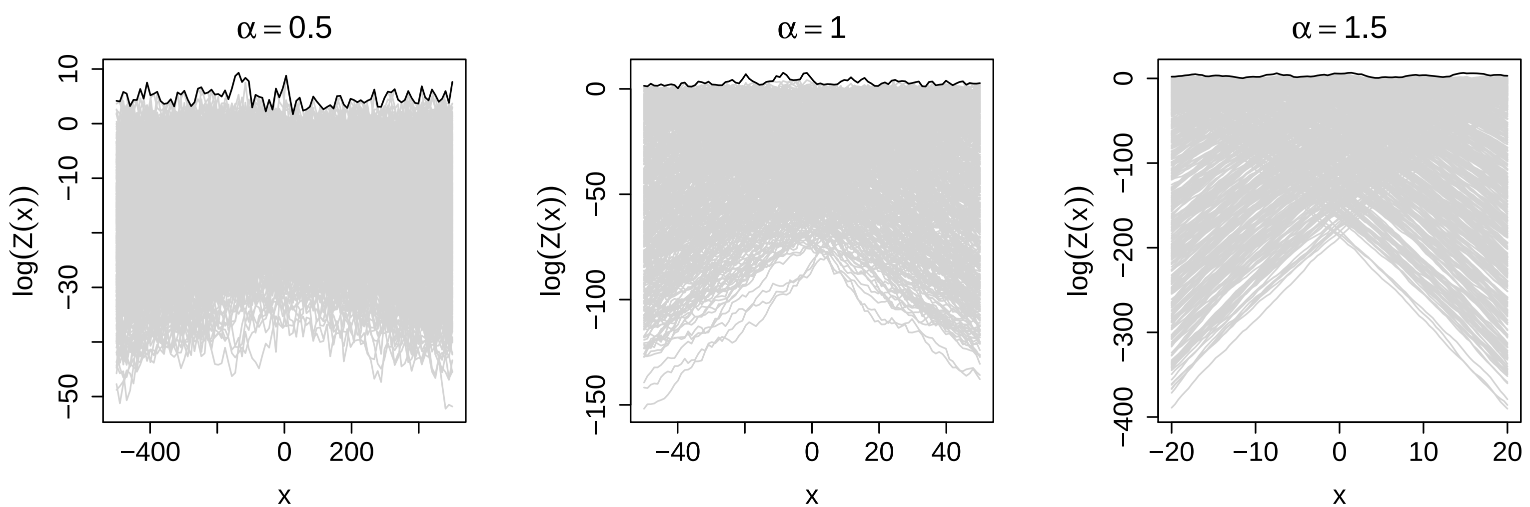

Figure 6: *Simulation of a Brown–Resnick process using uniform random shiftings.*

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Computational burden

- Let $\{Z(x)\}$ be a (unit Fréchet) max-stable process. We have

$$\Pr[Z(x_1) \leq z_1, \ldots, Z(x_K) \leq z_K] = \exp\{-V(z_1, \ldots, z_K)\}.$$

- The corresponding density is therefore

$$f(z_1, \ldots, z_K) = \frac{\partial^K}{\partial z_1 \cdots \partial z_K} \Pr[Z(x_1) \leq z_1, \ldots, Z(x_K) \leq z_K].$$

- When $K = 2$, $f = (V_1 V_2 - V_{12}) \exp(-V)$
- When $K = 3$, $f = (V_{12} V_3 + V_{13} V_2 + V_1 V_{23} - V_{123} - V_1 V_2 V_3) \exp(-V)$
- Combinatorial explosion: when $K = 10$ a single likelihood evaluation would require a sum of over 100,000 different terms.
- How to bypass this computational burden? Use pairwise likelihood.

Composite likelihoods

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Definition. Let $\{f(y;\theta), y \in \mathcal{Y}, \theta \in \Theta\}$ a parametric statistical model, where $\mathcal{Y} \subseteq \mathbb{R}^K$, $\Theta \subseteq \mathbb{R}^p$, $K \geq 1$ and $p \geq 1$.
Consider a set of (marginal or conditional) events $\{\mathcal{A}_i : \mathcal{A}_i \subseteq \mathcal{F}, i \in I\}$, where $I \subseteq \mathbb{N}$ and $\mathcal{F}$ is a σ-algebra on $\mathcal{Y}$.
A log-composite likelihood is defined as

$$\ell_c(\theta; y) = \sum_{i \in I} w_i \log f(y \in \mathcal{A}_i; \theta)$$

where $f(y \in \mathcal{A}_i; \theta) = f(\{y_j \in \mathcal{Y} : y_j \in \mathcal{A}_i\}; \theta)$, $y = (y_1, \ldots, y_n)$ and $\{w_i, i \in I\}$ is a set of suitable weights.

- In a nutshell, log-composite likelihoods are just linear combinations of (smaller) log-likelihood entities

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- First, note that the "full likelihood" is a special case of composite likelihood
- For i being fixed, $\log f(y \in \mathcal{A}_i; \theta)$ is a valid log-likelihood
- Thus leading to an unbiased estimating equation

$$\nabla \log f(y \in \mathcal{A}_i; \theta) = 0$$

- Finally $\nabla \ell_c(\theta; y) = \sum_{i \in I} w_i \nabla \log f(y \in \mathcal{A}_i; \theta) = 0$ is unbiased — as a linear combination of unbiased estimating equations
- For max-stable processes, as only the bivariate densities are known we will consider the pairwise likelihood

$$\ell_p(\mathbf{y}; \theta) = \sum_{k=1}^{n} \sum_{i<j} \log f(y_k^{(i)}, y_k^{(j)}; \theta)$$

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- Instead of having

$$\sqrt{n}\{H(\theta)\}^{1/2}(\hat{\theta}-\theta) \xrightarrow{d} N(\mathbf{0}, \mathrm{Id}_p)\,, \qquad n \to +\infty$$

where $H(\theta) = -\mathbb{E}[\nabla^2 \ell(\theta; \mathbf{Y})]$, $(M^{1/2})^T M^{1/2} = M$

- When we work under misspecification - which is the case when using composite likelihoods, we now have

$$\sqrt{n}\{H(\theta)J(\theta)^{-1}H(\theta)\}^{1/2}(\hat{\theta}-\theta) \xrightarrow{d} N(\mathbf{0}, \mathrm{Id}_p)\,, \qquad n \to +\infty$$

where $J(\theta) = \mathrm{Var}[\nabla \ell(\theta; \mathbf{Y})]$

- Note that if the 2nd Bartlett idendity holds then

$$H(\theta)J(\theta)^{-1}H(\theta) = H(\theta),$$

i.e., usual MLE asymptotics.

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- Since we use composite likelihood, standard tools for model selection cannot be used
- However IC and likelihood ratio tests can be used up to slight modifications, i.e.,

$$\text{TIC} = -2\ell_p(\hat{\theta}) + k\,\mathrm{tr}\{J(\hat{\theta})^{-1}H(\hat{\theta})\}, \qquad k = 2, \log n$$

and

$$2\left\{\ell_p(\hat{\theta}) - \ell_p(\hat{\psi}, \gamma_0)\right\} \xrightarrow{d} \sum_{i=1}^{p} \lambda_i X_i, \qquad n \to \infty,$$

where $X_i \overset{iid}{\sim} \chi^2_1$.

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Table 1: *Summary of the max-stable models fitted to the Swiss rainfall data. Standard errors are in parentheses.* $(*)$ *denotes that the parameter was held fixed.* h_- *and* h_+ *are respectively the distances for which* $\theta(x)$ *is equal to* 1.3 *and* 1.7*. NoP is the number of parameters.* ℓ_p *is the pairwise log-likelihood evaluated at its maximum. TIC is an information criterion for model selection under misspecification —* $TIC = -2\ell_p + 2\text{tr}\{JH^{-1}\}$.

Smith								
Correlation	σ_{11}	σ_{12}	σ_{22}	h_-	h_+	NoP	ℓ_p	TIC
Isotropic	259 (45)	0 ($*$)	$\sigma_{22} = \sigma_{11}$	12.4	33	8	−212455	427113
Anisotropic	251 (46)	64 (13)	290 (50)	6.6–11.1	18–30	10	−212395	427020
Schlather								
Correlation		λ	κ	h_-	h_+	NoP	ℓ_p	TIC
Whittle		39.3 (21.4)	0.44 (0.12)	6.0	147	9	−210813	424200
Cauchy		8.0 (2.2)	0.34 (0.16)	7.1	2370	9	−210874	424321
Stable		34.8 (11.5)	0.95 (0.16)	6.3	146	9	−210815	424206
Exponential		34.1 (9.0)	——	6.8	134	8	−210816	424167
Geometric Gaussian								
Correlation	σ^2	λ	κ	h_-	h_+	NoP	ℓ_p	TIC
Whittle	11.05 (3.84)	700 ($*$)	0.37 (0.03)	5.8	86	9	−210349	423232
Cauchy	30.85 (8.14)	5.21 (0.66)	0.01 ($*$)	6.7	192	9	−210412	423355
Stable	15.04 (5.36)	1000 ($*$)	0.76 (0.06)	5.9	86	9	−210349	423233
Exponential	2.42 (0.93)	53.2 (18.4)	——	7.0	116	9	−210368	423271
Brown–Resnick								
Correlation		λ	α	h_-	h_+	NoP	ℓ_p	TIC
fBm		30 (9.23)	0.74 (0.07)	5.8	84	9	−210348	423231

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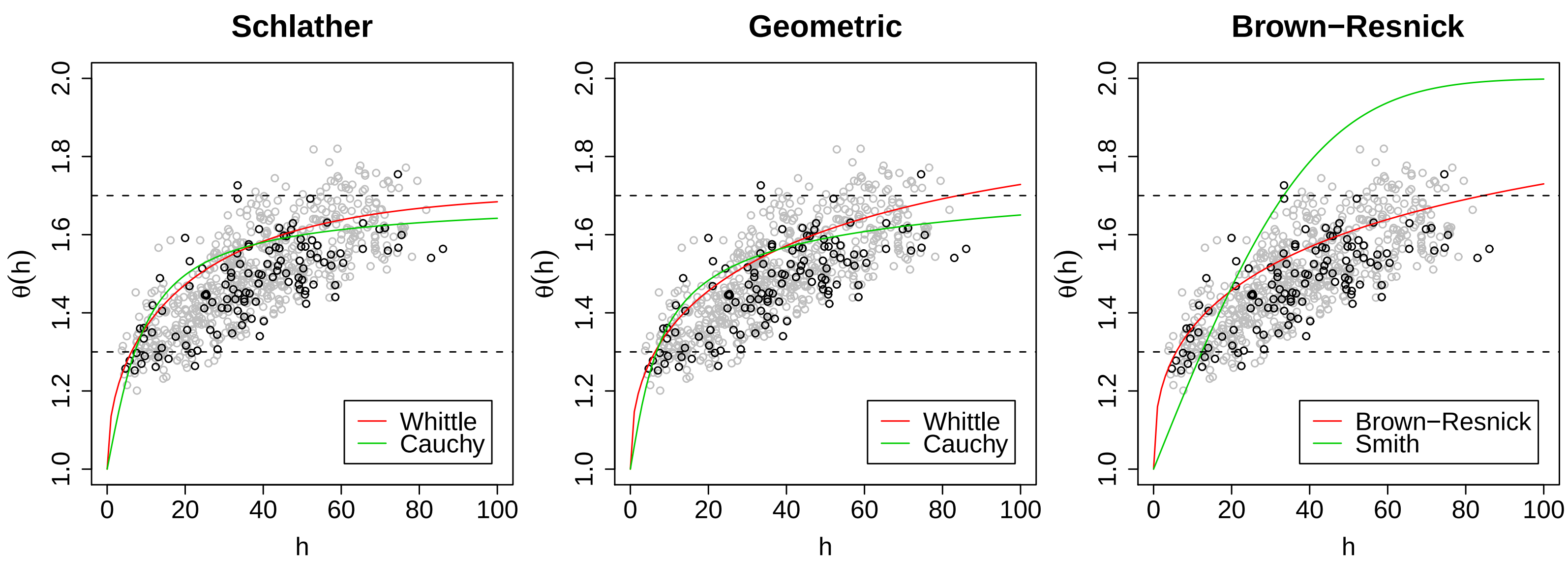

Figure 7: *Comparison between the F-madogram estimates for the fitting (grey points) and the validation (black points) data sets and the estimated extremal coefficient functions for different max-stable models.*

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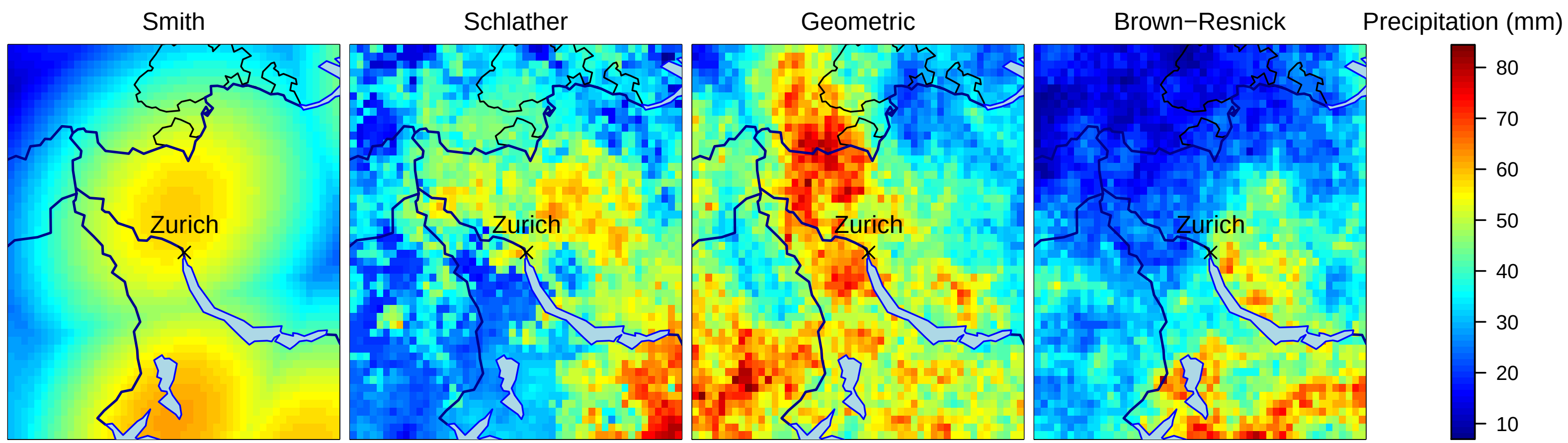

Figure 7: *One realization of the best Smith, Schlather, geometric Gaussian and Brown–Resnick max-stable models, on a* 50×50 *grid.*

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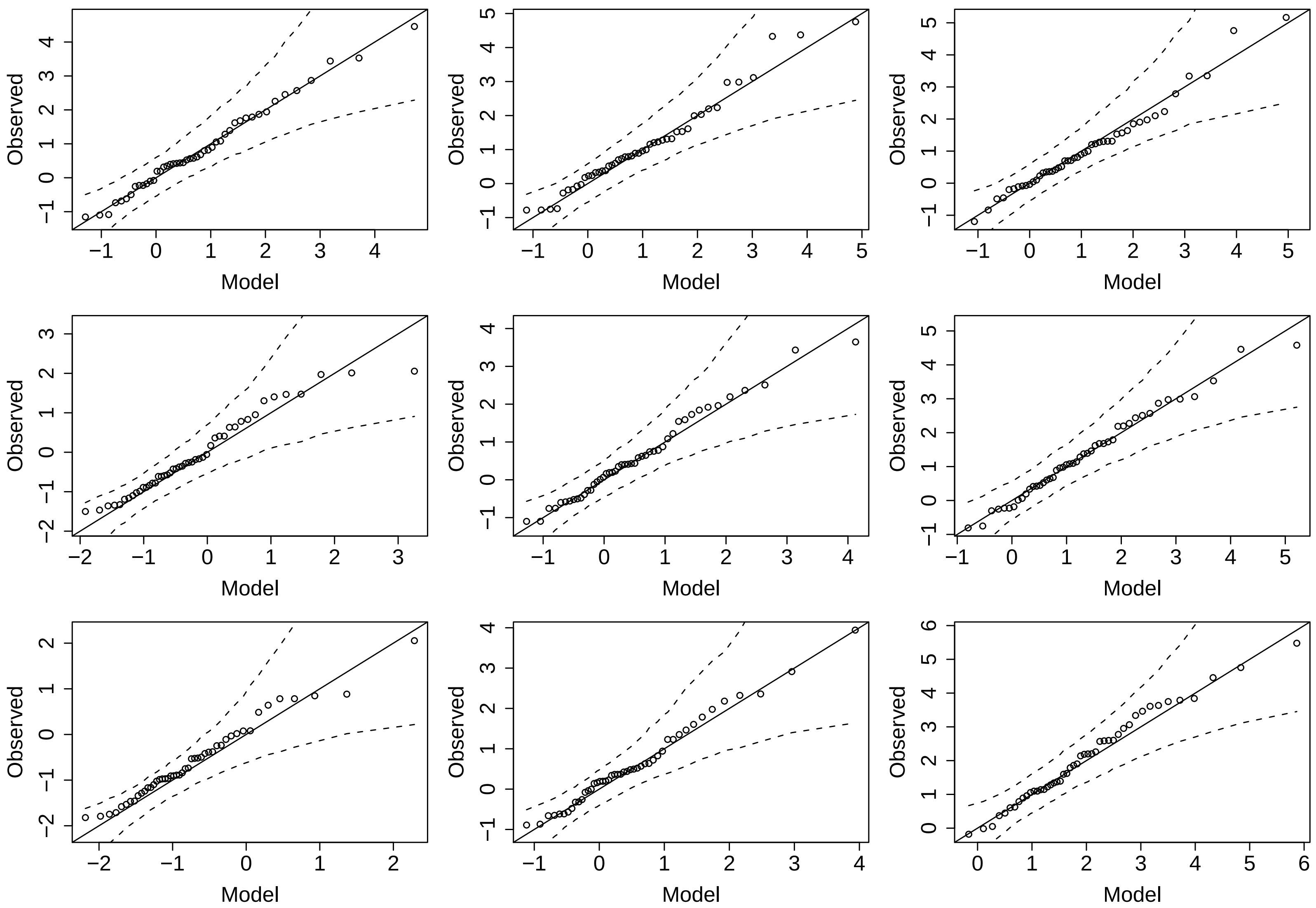

Figure 7: *Model checking for the best max-stable model (Brown–Resnick).*

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- The modelling of spatial extremes is not simple — really!
- Max-stable processes are the natural extension of the EVT to the infinite dimensional setting
- But these are difficult to simulate from and to fit to spatial data
- The deterministic trend surfaces might be too smooth to be realistic depending on the available covariates
- Embedding max-stable processes into a Bayesian hierarchical model is promising — but further theory for non standard Bayesian inference is required.
- Next step: conditional simulations ?

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Thank you for your attention!

Advertising:

- If you want to play with max-stable processes, have a look at the `SpatialExtremes` R package

  `http://spatialextremes.r-forge.r-project.org/`

- This talk was based on

  Davison, A.C., Padoan, S.A. and Ribatet, M.
  Statistical Modelling of Spatial Extremes.
  To appear in Statistical Science.