# Likelihood-based inferences for max-stable processes

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# Spectral Representation of Max-stable Processes

**Theorem (de Haan (1987))**

*Let $\{\xi_i\}_{i\geq 1}$ be the points of a homogeneous Poisson process on $\mathbb{R}_+$ with intensity $d\Lambda(\xi) = \xi^{-2}d\xi$, and $\{Y_i(\cdot)\}_{i\geq 1}$ be i.i.d. replicates of a stationary process on $\mathbb{R}^d$ such that $\mathbb{E}[\max\{0, Y(x)\}] = 1$.*
*Then*

$$Z(x) = \max_i \xi_i \max\{0, Y_i(x)\}$$

*is a stationary max-stable process with unit Fréchet margins*

- Different choices for $Y(\cdot)$ lead to different max-stable processes
- Max-stable processes are asymptotically justified models for modelling spatial extremes
- But inferential procedure for such processes are at an early stage

# Parametric max-stable models (1)

## Smith [1990]

Let $Y_i(x) = \varphi(x - X_i)$ where $\{X_i\}$ is a homogeneous Poisson process and $\varphi$ is the zero mean multivariate normal density with covariance matrix $\Sigma$, both on $\mathbb{R}^d$. Then

$$\Pr[Z(x_1) \leq z_1, Z(x_2) \leq z_2] = \exp\left[-\frac{1}{z_1}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_2}{z_1}\right) - \frac{1}{z_2}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_1}{z_2}\right)\right]$$

where $\Phi$ is the standard normal CDF and $a^2 = \Delta x^T \Sigma^{-1} \Delta x$.

## Schlather [2002]

Let $Y_i(\cdot) \sim GP(\mu, \rho)$ scaled such that $\mathbb{E}[\max\{0, Y_i(x)\}] = 1$. Then

$$\Pr[Z(x_1) \leq z_1, Z(x_2) \leq z_2] = \exp\left[-\frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\left(1 + \sqrt{1 - 2(\rho(h) + 1)\frac{z_1 z_2}{(z_1 + z_2)^2}}\right)\right]$$

where $h = ||x_1 - x_2||$.

# Parametric max-stable models (2)

## Geometric Gaussian Model [A.C.D.]

Let $Y_i(x) = \exp\{\sigma\epsilon(x) - \sigma^2/2\}$ where $\epsilon(\cdot)$ is a standard gaussian process. Note that by definition, $Y_i(x) > 0$ and $\mathbb{E}[Y(x)] = 1$. Then the bivariate CDF is the same as for the Smith model where

$$a^2 = 2\sigma^2\{1 - \rho(h)\}$$

It is possible to generalize this model i.e. $\sigma(x)$

## Kabluchko et al. [2009]

Let $Y_i(x) = \exp\{\epsilon(x) - \sigma^2(x)/2\}$ where $\epsilon(\cdot)$ is a Gaussian process with stationary increments and $\sigma^2(x) = \text{Var}[\epsilon(x)]$. Then the bivariate CDF is the same as for the Smith model where

$$a^2 = \gamma(x_2 - x_1)$$

where $\gamma(\cdot)$ is the variogram of $\epsilon(\cdot)$.

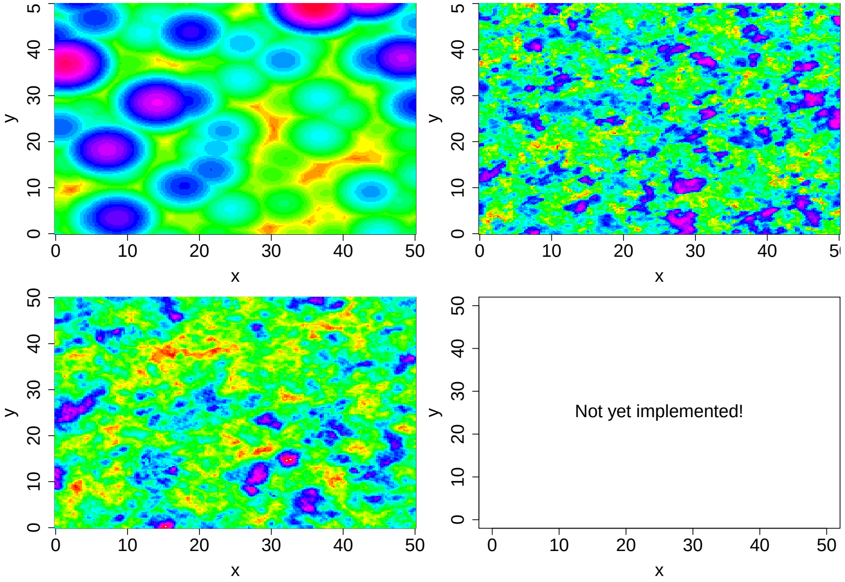



Figure: Top left: Smith. Top right: Schlather. Bottom left: Geometric. Bottom right: Brown–Resnick. Gumbel margins.

# (Pairwise) Extremal coefficient function

- W.l.o.g., if we suppose unit Fréchet margins

$$\Pr\left[Z(x_1) \le z, Z(x_2) \le z\right] = \exp\left(-\frac{\theta(x_2 - x_1)}{z}\right)$$

  where $1 \le \theta(x_2 - x_1) \le 2$
- $\theta(x_2 - x_1) = 1 \iff$ perfect dependence
- $\theta(x_2 - x_1) = 2 \iff$ independence
- This provides information about the dependence between locations $x_1$ and $x_2$

**Smith** $\theta(x_2 - x_1) = 2\Phi\left(\frac{\sqrt{(x_1-x_2)^T\Sigma^{-1}(x_1-x_2)}}{2}\right)$

**Schlather** $\theta(x_1 - x_2) = 1 + \sqrt{\frac{1-\rho(x_1-x_2)}{2}}$

**Geometric Gaussian** $\theta(x_1 - x_2) = 2\Phi\left(\sqrt{\frac{\sigma^2(1-\rho(x_1-x_2))}{2}}\right)$

**Brown–Resnick** $\theta(x_1 - x_2) = 2\Phi\left(\frac{\sqrt{\gamma(x_1-x_2)}}{2}\right)$

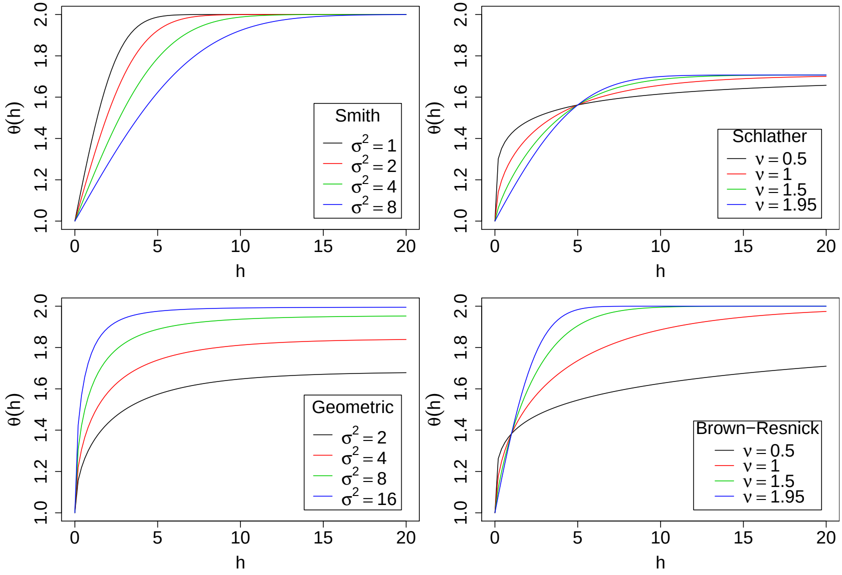



Figure: Examples of extremal coefficient functions for the models introduced.

# Composite likelihood

- Only the bivariate CDF are analytically known
- MLE is therefore hopeless
- But one can work with composite likelihood

**Definition**

Let $\{f(y;\theta), y \in \mathcal{Y}, \theta \in \Theta\}$ a parametric statistical model, where $\mathcal{Y} \subseteq \mathbb{R}^n$, $\Theta \subseteq \mathbb{R}^d$, $n \geq 1$ and $d \geq 1$.
Consider a set of events $\{\mathcal{A}_i : \mathcal{A}_i \subseteq \mathcal{F}, i \in I\}$, where $I \subseteq \mathbb{N}$ and $\mathcal{F}$ is a $\sigma$-algebra on $\mathcal{Y}$.
A log-composite likelihood is defined as

$$\ell_c(\theta; y) = \sum_{i \in I} w_i \log f(y \in \mathcal{A}_i; \theta)$$

where $f(y \in \mathcal{A}_i; \theta) = f(\{y_j \in \mathcal{Y} : y_j \in \mathcal{A}_i\}; \theta)$, $y = (y_1, \ldots, y_n)$ and $\{w_i, i \in I\}$ is a set of suitable weights.

# Why does it works?

- First, note that the "full likelihood" is a special case of composite likelihood
- For $i$ being fixed, $\log f(y \in \mathcal{A}_i; \theta)$ is a valid log-likelihood
- Thus leading to an unbiased estimating equation

$$\nabla \log f(y \in \mathcal{A}_i; \theta) = 0$$

- Finally $\nabla \ell_c(\theta; y) = \sum_{i \in I} w_i \nabla \log f(y \in \mathcal{A}_i; \theta) = 0$ is unbiased - as a linear combination of unbiased estimating equations
- For max-stable processes, as only the bivariate densities are known we will consider the pairwise likelihood

$$\ell_p(\mathbf{y}; \psi) = \sum_{i<j} \sum_{k=1}^{n} \log f(y_k^{(i)}, y_k^{(j)})$$

# Asymptotics

- Instead of having

$$\sqrt{n}(\hat{\psi} - \psi) \xrightarrow{D} N\big(\mathbf{0}, -H(\psi)^{-1}\big)\,, \qquad n \to +\infty$$

where $H(\psi) = \mathbb{E}[\nabla^2 \ell(\psi; \mathbf{Y})]$

- When we work under misspecification - which is the case when using composite likelihoods, we now have

$$\sqrt{n}(\hat{\psi} - \psi) \xrightarrow{D} N\big(\mathbf{0}, H(\psi)^{-1} J(\psi) H(\psi)^{-1}\big)\,, \qquad n \to +\infty$$

where $J(\psi) = \mathrm{Var}[\nabla \ell(\psi; \mathbf{Y})]$

- Note that when the model is correctly specified, $H(\psi) = -J(\psi)$ and $H(\psi)^{-1} J(\psi) H(\psi)^{-1} = -H(\psi)^{-1}$

# Simulation Study: MPLE Performance

- Spatial domain: $\varkappa = [0, 40] \times [0, 40]$
- Data: 50 sites and 100 obs./site
- 500 replications of the experiment: Smith model, with 5 different $\Sigma$ matrices

| | $\sigma_1^2$ | $\sigma_{12}$ | $\sigma_2^2$ | Dependence |
|---|---|---|---|---|
| Conf.1 | 300 | 0 | 300 | isotropy |
| Conf.2 | 200 | 0 | 300 | anisotropy |
| Conf.3 | 200 | 150 | 300 | medium |
| Conf.4 | 2000 | 1500 | 3000 | strong |
| Conf.5 | 20 | 15 | 30 | weak |

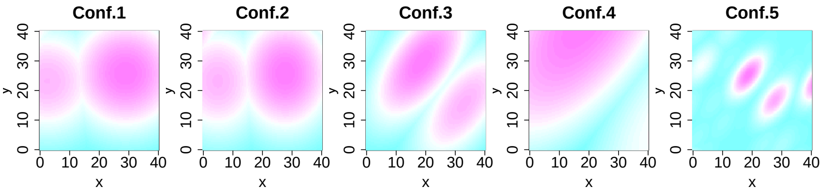



Figure: One realisation of the Smith model for each covariance matrice.

# Results

Table: MPLE performance. Are displayed: $\frac{1}{n}\sum_{i=1}^{n}\hat{\psi}_{p,i}$ (theo. val.) / std. err. sandwich (emp. std. err.)

| | $\hat{\sigma}_1^2$ / std. err. | $\hat{\sigma}_2^2$ / std. err. | $\hat{\sigma}_{12}^2$ / std. err. |
|---|---|---|---|
| Conf.1 | 306.13 (300) / 40.59 (44.70) | 305.74 (300) / 39.80 (41.54) | 1.35 (0) / 27.91 (27.74) |
| Conf.2 | 203.95 (200) / 26.70 (28.54) | 305.35 (300) / 39.55 (39.66) | -0.95 (0) / 21.92 (21.23) |
| Conf.3 | 201.84 (200) / 25.09 (26.10) | 299.53 (300) / 37.34 (37.88) | 150.01 (150) / 25.53 (26.13) |
| Conf.4 | 2053.37 (2000) / 495.22 (300.10) | 3065.76 (3000) / 664.79 (483.11) | 1550.15 (1500) / 412.00 (322.37) |
| Conf.5 | 19.99 (20) / 1.53 (1.55) | 29.89 (30) / 2.30 (2.29) | 14.95 (15) / 1.55 (1.60) |

- Only a small bias on the estimation of $\theta$
- Std. err. from the sandwich covariance matrix are consistent with their empirical counterparts
- Std. err : What's wrong with Conf.4?
  Break in regularity conditions? $\ell_c(\hat{\psi})$ too wiggly?

$$\Sigma = \begin{bmatrix} 2000 & 1500 \\ 1500 & 3000 \end{bmatrix} \Longrightarrow \lambda_{1,2}(\Sigma^{-1}) = \frac{5 \pm \sqrt{10}}{7500} \approx 10^{-3}, 10^{-4}$$

  Unreliable estimation of $H$? i.e. finite differences

# Information criteria

- When several models M0, M1, . . . are fitted to our data, one would prefer the one minimising

$$AIC = -2\ell(\hat{\theta}_{MLE}; y) + 2p$$

where $p$ is the number of parameters to be estimated.

- Under misspecification, one should use

$$TIC = -2\ell(\hat{\theta}; y) - 2\text{tr}\left\{J(\psi)H(\psi)^{-1}\right\}$$

as the 2nd Bartlett idendity is not true anymore

- Note that if the model is correctly specified

$$J(\psi) = -H(\psi), \qquad \mathbb{E}\left[\nabla^2\ell(\theta; y)\right] + \text{Var}\left[\nabla\ell(\theta; y)\right] = 0$$

so that

$$TIC = -2\ell(\hat{\theta}; y) + 2\text{tr}\left\{\mathbb{I}_p\right\} = AIC$$

# Likelihood ratio statistic

- Let $\{f(y;\theta), y \in \mathcal{Y}, \theta \in \Theta\}$ our statistical model
- Suppose that $\theta^T = (\psi^T, \phi^T)$ where $\psi$ and $\phi$ are vector of dimension $p$ and $q$.
- $H_0 : \psi = \psi_0 \quad \text{vs.} \quad H_1 : \psi \neq \psi_0$
- When the model is correctly specified,

$$W(\psi_0) = 2\{\ell(\hat{\theta}; y) - \ell(\psi_0, \hat{\phi}_{\psi_0}; y)\} \xrightarrow{D} \chi^2_p$$

- Under misspecification, this results is slightly modified

$$W(\psi_0) = 2\{\ell_c(\hat{\theta}; y) - \ell_c(\psi_0, \hat{\phi}_{\psi_0}; y)\} \xrightarrow{D} \eta = \sum_{i=1}^{p} \lambda_i X_i$$

  where $X_i \overset{iid}{\sim} \chi^2_1$ and the $\lambda_i$ are the eigenvalues of $(H^{-1}JH^{-1})_\psi \left\{-(H^{-1})_\psi\right\}^{-1}$, $M_\psi = M["\psi", "\psi"]$
- Note that if the model is correctly specified, $(H^{-1}JH^{-1})_\psi \left\{-(H^{-1})_\psi\right\}^{-1} = \mathbb{I}_p$

- Unfortunately the distribution of $\eta = \sum_{i=1}^{p} \lambda_i X_i$ is not known
- Two different approaches are possible :
  - Approximate the distribution of $\eta$ [Rotnitzky and Jewell, 1990]

$$\eta \approx \sum_{i=1}^{p} \hat{\lambda}_i X_i$$

  - Adjust $W(\psi_0)$ in such a way that it still converges to the usual $\chi^2_p$ [Chandler and Bate, 2007]

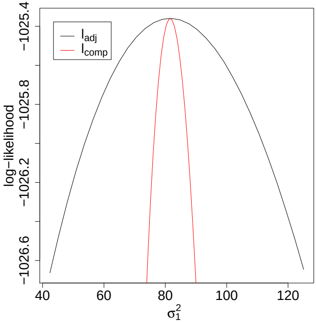
$$W_{adj}(\psi_0) = 2c \left\{ \ell_{adj}(\hat{\theta}; y) - \ell_{adj}(\psi_0, \hat{\phi}_{\psi_0}; y) \right\}$$

where $c$ is a quadratic approximation and

$$\ell_{adj}(\theta) = \ell_c(\theta_*), \qquad \theta_* = \hat{\theta} + M^{-1} M_{adj}(\theta - \hat{\theta})$$

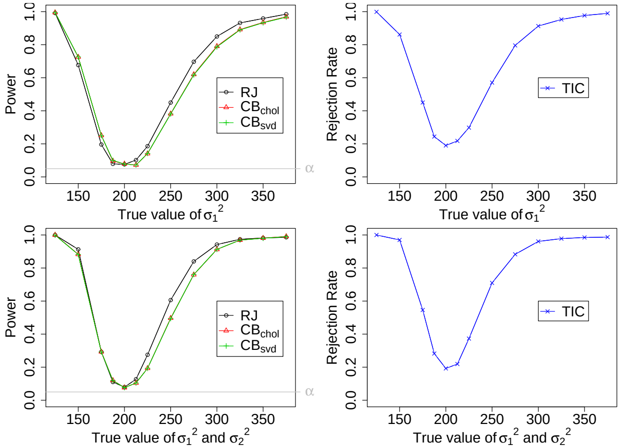
where $M^T M = H$ and $M_{adj}^T M_{adj} = H^{-1} J H^{-1}$.

- $c$ is needed as most often $\hat{\phi}_{\psi_0}$ will be obtained by maximising $\ell_p$ and not $\ell_{adj}$
- Hence $\ell_{adj}(\psi_0, \hat{\phi}_{\psi_0}) \leq \ell_{adj}(\psi_0, \hat{\phi}^{adj}_{\psi_0})$
- Resulting in liberal tests of hypotheses



# Power of the likelihood ratio test and TIC rejection rates

- Spatial domain : $\varkappa = [0, 40] \times [0, 40]$
- Data: 50 sites and 100 obs./site
- Model: Smith $\Sigma = \begin{bmatrix} \sigma_1^2 & 150 \\ 150 & 300 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1^2 & 150 \\ 150 & \sigma_2^2 \end{bmatrix}$
- $H_0 : \sigma_1^2 = 200 \ (= \sigma_2^2)$ vs. $H_1$ : compl.



# Switching to ordinary GEV margins

- Consider the mapping $t$ that transform GEV obs. to a unit Fréchet scale

$$t : Y(x) \mapsto \left(1 + \xi(x)\frac{Y(x) - \mu(x)}{\sigma(x)}\right)^{1/\xi(x)}$$

- Hence the bivariate distribution might be rewritten as

$$\Pr[Y(x_1) \leq y_1, Y(x_2) \leq y_2] = \Pr[Z(x_1) \leq t(y_1), Z(x_2) \leq t(z_2)]$$

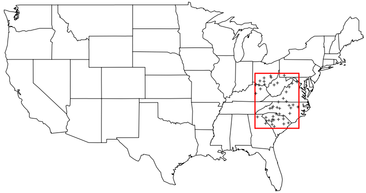
  where $Z(\cdot)$ is a unit Fréchet max-stable process
- The log-likelihood becomes

$$\ell_p(\mathbf{y}; \psi) = \sum_{i<j} \sum_{k=1}^{n} \left\{\log f\{t(z_k^{(i)}), t(z_k^{(j)}); \psi\} + \log |J(y_k^{(i)}) J(y_k^{(j)})|\right\}$$

  where $|J(t(y_k^{(i)}))|$ is the jacobian of the mapping $t$

# Extreme precipitations in the US

- 46 stations (91 obs./stations)
- $\varkappa = \mathbb{R}^2$, alt add. cov.
- Smith's model (anisotropy)



| Models | | $-\ell_p$ | Dof | TIC |
|---|---|---|---|---|
| $M_0$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat}) + \alpha_2(\text{alt}) + \alpha_3(\text{lon})$ | 412110.2 | 12 | 825679 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_2(\text{alt}) + \beta_2(\text{lon})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |
| $M_1$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat}) + \alpha_2(\text{alt})$ | 412111.7 | 11 | 825526 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_2(\text{alt}) + \beta_3(\text{lon})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |
| $M_2$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat}) + \alpha_2(\text{alt}) + \alpha_3(\text{lon})$ | 412,113.6 | 11 | 825459 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_2(\text{alt})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |
| $M_3$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat}) + \alpha_3(\text{lon})$ | 412234.4 | 11 | 825,840 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_2(\text{alt}) + \beta_3(\text{lon})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |
| $M_4$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat}) + \alpha_2(\text{alt}) + \alpha_3(\text{lon})$ | 412380.9 | 11 | 826177 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_3(\text{lon})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |
| $M_5$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat}) + \alpha_2(\text{alt})$ | 412113.9 | 10 | 825327 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_2(\text{alt})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |
| $M_6$ : | $\mu(x) = \alpha_0 + \alpha_1(\text{lat})$ | 412314.4 | 9 | 825684 |
| | $\sigma(x) = \beta_0 + \beta_1(\text{lat}) + \beta_2(\text{alt})$ | | | |
| | $\xi(x) = \gamma_0$ | | | |

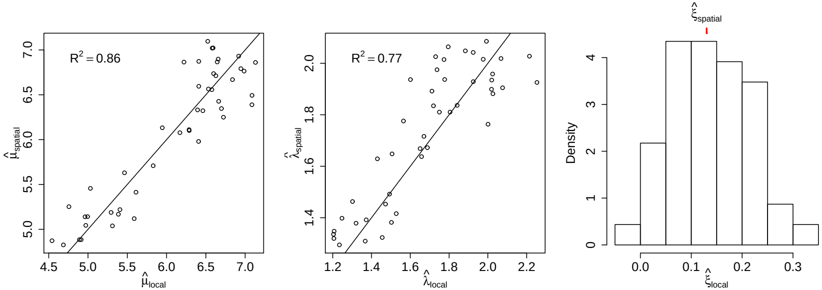



Figure: GEV parameters estimated locally (MLE) and from the fitted max-stable model.

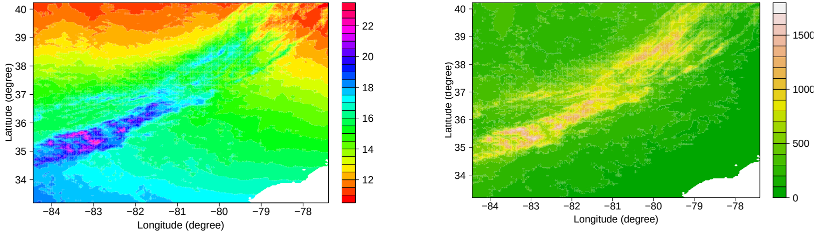



Figure: Pointwise return levels ($T = 50$ ans, cm) (left) and the elevation map (right, meter)

- Consider $Z_i = \max\{Y_{i,j}, Y_{i,k}\}$, $i = 1, \ldots, n$
- Compare observed and simulated $\{Z_i\}_{i=1}^{n}$

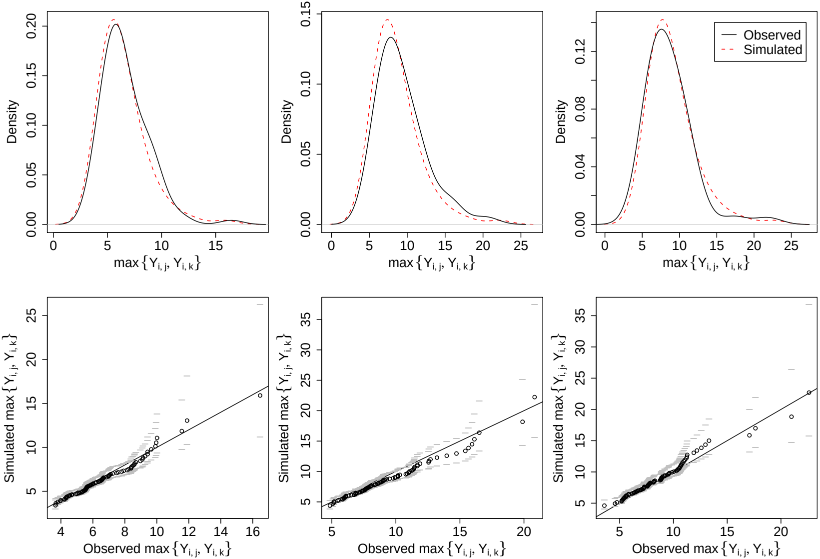



Figure: Left: short distance (≈ 20 km). Middle: medium distance (≈ 350 km). Right: long distance (≈ 735 km).

# Extremal coefficient contours and conditional quantiles

- One can defined conditional return levels i.e.

$$\Pr[Z(x_2) > z_2 | Z(x_1) > z_1] = \frac{1}{T_2}$$

  where $\Pr[Z(x_1) \leq z_1] = 1 - 1/T_1$
- This is the level which is expected to be exceeded once every $T_2$ year, given that at $x_1$ we exceed the level $z_1$

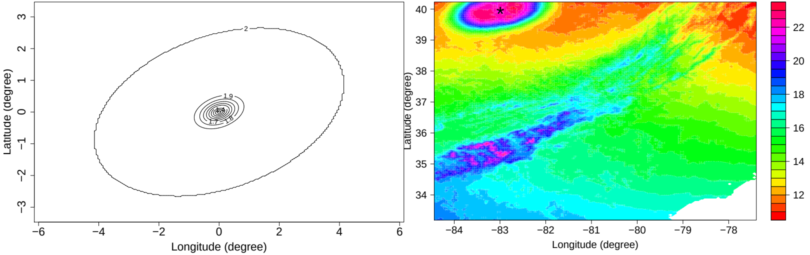



Figure: Evolution of $\theta(x_2 - x_1)$ in $\mathbb{R}^2$ (degrees) and conditional return levels ($T_1 = T_2 = 50$ years)

# Using weights in the pairwise likelihood

- We used $\ell_p(\theta) = \sum_{i<j} \sum_k \omega_{i,j} \log f(y_k^{(i)}, y_k^{(j)}; \theta)$, $\omega_{i,j} \equiv 1$
- Following ideas on conventional geostatistics, one can use

$$\omega_{i,j} = 1, \text{ if } ||x_i - x_j|| \leq \delta, \qquad \omega_{i,j} = 0, \text{ otherwise}$$

- Optimal $\delta_* = \underset{\delta}{\text{argmin}}\ f(H^{-1}JH^{-1})$ for some cost function $f$

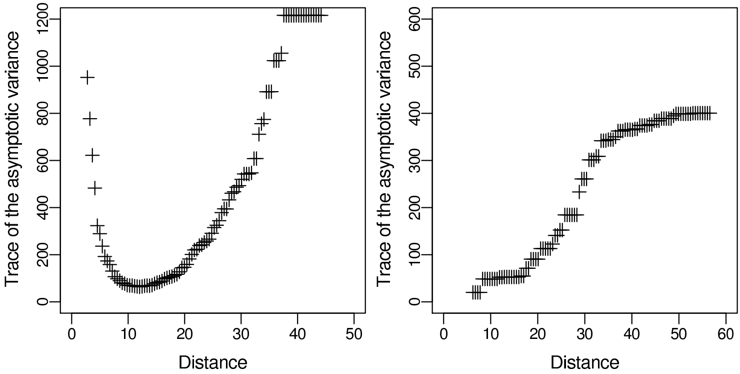



Figure: Trace of $H^{-1}JH^{-1}$. Left: scattered locations. Right: gridded locations.

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# Thank you for your attention!

This work has been done by using the R package *SpatialExtremes*

`http://spatialextremes.r-forge.r-project.org/`