

Functional regular variations, Pareto processes and peaks over threshold

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Abstract

Although the last decades have seen many developments on max-stable processes, little is known on the limiting distribution of exceedances of stochastic processes. Paralleling the univariate extreme value theory, this work focuses on threshold exceedances of a stochastic process and their connections with regularly varying and generalized Pareto processes. More precisely we define an exceedance through a cost functional ℓ and show that the limiting (rescaled) distribution is a simple ℓ -Pareto process whose spectral measure can be characterized. Several equivalent constructions for ℓ -Pareto processes are given using either a constructive approach, either an homogeneity property or a peak over threshold stability. We also provide an estimator of the spectral measure and give some examples.

Key words: Extreme value theory, functional regular variations, generalized Pareto process, Peaks Over Threshold

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1 Introduction

Balkema and de Haan [4] and Pickands [16] have made a major contribution to the extreme value theory with the introduction of the generalized Pareto distribution and its connection with exceedances above a large threshold. They established that the linearly normalized maximum of independent random variables converges to an extreme value distribution if and only if the normalized exceedance above a threshold converges to a generalized Pareto distribution. For statistical purposes, the use of peaks over threshold rather than block maxima is often more convenient since it usually wastes less observations. Extensions to the multivariate case have been proposed by Rootzen and Tajvidi [17] and Falk et al. [12].

More recently, the infinite dimensional setting, i.e., the functional framework and continuous random processes, enjoyed renewed interests. The generalized Pareto processes, also known as GPD processes or functional generalized Pareto distributions, have been introduced by Buishand et al. [5], Aulbach et al. [3] and de Haan and Ferreira [8]. Similarly to the finite dimensional case, the domain of attraction of a generalized Pareto process and that of the associated max-stable process coincide. Several equivalent characterizations of Pareto processes are given, including the peak over threshold stability and a homogeneity property. Statistical issues such as local asymptotic normality or tests for the class of generalized Pareto processes are addressed in Aulbach and Falk [1, 2].

Often exceedances above a high threshold can be defined through a uniform supremum. More precisely, a peak over threshold of a stochastic process $\{X(t)\}_{t \in T}$ can be defined by

$$\sup_{t \in T} \frac{X(t) - b_n(t)}{a_n(t)} > 0$$

where $\{a_n > 0\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ are normalizing functions. Since we restrict our

attention to the tails of the process, it is sensible to have

$$\mathbb{P} \left\{ \sup_{t \in T} \frac{X(t) - b_n(t)}{a_n(t)} > 0 \right\} \longrightarrow 0, \quad n \rightarrow \infty.$$

Theorem 4.2 in de Haan and Ferreira [8] states that if there exists continuous normalizing functions $\{a_n > 0\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ such that

$$\frac{X - b_n}{a_n} \mid \left\{ \sup_{t \in T} \frac{X(t) - b_n(t)}{a_n(t)} > 0 \right\}$$

converges weakly in the space of continuous functions as $n \rightarrow \infty$, then the limit must be a generalized Pareto process. In particular when X is nonnegative and $a_n = b_n > 0$ are constants, we have

$$\sup_{t \in T} \frac{X(t) - b_n}{a_n} > 0 \quad \text{if and only if} \quad \|X\| > a_n$$

where $\|\cdot\|$ denotes the uniform norm.

Although from a theoretical point of view the use of a uniform supremum seems sensible when working with the space of continuous functions, for practical purposes other kinds of thresholds might be relevant. More generally, an exceedance over a threshold can be defined as an event $\{\ell(X) > a_n\}$, for some functional ℓ and where the threshold a_n is such that

$$\mathbb{P} \{\ell(X) > a_n\} \longrightarrow 0, \quad n \rightarrow \infty.$$

For example, Buishand et al. [5] were interested in the total amount of rain over a catchment T , i.e., $\int_{t \in T} X(t) dt$ where $X(t)$ represents the amount of rain at $t \in T$. In this context, it seems appropriate to let $\ell(x) = \int_{t \in T} x(t) dt$ and to derive the limiting distribution of

$$a_n^{-1} X \mid \{\ell(X) > a_n\}, \quad n \rightarrow \infty.$$

Other possibilities are $\ell_1(x) = \int_T |x(t)|^2 dt$, $\ell_2(x) = \inf_{t \in T} x(t)$ or $\ell_3(x) = x(t_0)$. The choice ℓ_1 is natural in the context of an energy functional, for example if X stands for the strength of the wind in space. A high threshold with respect to ℓ_2 occurs when the random field takes large values at any point $t \in T$ and might be relevant for modeling sea levels along a dike. The use of ℓ_3 puts the emphasis on a specific point $t_0 \in T$ and might be of interest for modeling extreme flows at the confluence of two rivers. We may see these functionals as *cost functionals* and for our purposes we will restrict our attention to the class of continuous (with respect to the uniform norm) nonnegative and homogeneous functional $\ell : \mathcal{C} \rightarrow [0, +\infty)$, with $\mathcal{C} = \mathcal{C}\{T, [0, +\infty)\}$ the Banach space of nonnegative continuous functions over a compact parameter set T . All the previous examples belong to this class. A functional ℓ is said to be homogeneous of order $\beta > 0$ if

$$\ell(ux) = u^\beta \ell(x), \quad u > 0, x \in \mathcal{C}.$$

Without loss of generality, we can assume that $\beta = 1$. Indeed since the functional ℓ is nonnegative, the functional $\tilde{\ell} = \ell^{1/\beta}$ is clearly homogeneous of order 1 and satisfies

$$\{\ell(X) > a_n\} = \{\tilde{\ell}(X) > a_n^{1/\beta}\}.$$

In this paper, we focus on functional extreme value theory in the space \mathcal{C} . The theory was initiated by de Haan [7] and de Haan and Pickands [11]. For a background on functional extreme value theory, we refer to de Haan and Ferreira [9]. Connections with functional regular variations are well known, see for example de Haan and Lin [10], Hult and Lindskog [13, 14] or Davis and Mikosch [6]. For the sake of simplicity, we will restrict our attention to standardized (or simple) processes, i.e., processes whose marginal distributions are in the domain of attraction of a Fréchet distribution.

2 Preliminaries

We introduce the framework of our approach and, following Davis and Mikosch [6], review some details on functional extreme value theory and regular variation theory. Other relevant references on this topic are [10, 13, 14].

We start with some standard results on univariate extreme value theory. Let $\alpha > 0$ and $\{X_i\}_{i \geq 1}$ an i.i.d. sequence of positive random variables with common distribution function F . For $t > 1$, we note $a(t) = F^{\leftarrow}(1 - 1/t)$, where F^{\leftarrow} denotes the quantile function. It is well known that the following statements are equivalent.

- i) the tail function $1 - F$ is regularly varying at infinity with index $-\alpha$;
- ii) $t\mathbb{P}\{X_1/a(t) \in \cdot\} \xrightarrow{v} \alpha u^{-\alpha-1}du$ as $t \rightarrow +\infty$, where \xrightarrow{v} stands for vague convergence in the space $M\{(0, +\infty]\}$ of Radon measure on $(0, +\infty]$;
- iii) the normalized sample point process $\sum_{i=1}^n \delta_{X_i/a(n)}$ converges weakly in $M\{(0, +\infty]\}$ to a Poisson point process on $(0, +\infty]$ with intensity $\alpha u^{-\alpha-1}du$;
- iv) the normalized maximum $\max(X_1, \dots, X_n)/a(n)$ converges in distribution to an α -Fréchet distribution;
- v) the distribution $\mathbb{P}\{X_1/a(t) \in \cdot \mid X_1 > a(t)\}$ of normalized exceedances over high threshold converges to a Pareto distribution with index α ;

We explain how this can be generalized to the functional setting and introduce first the notion of functional regular variation. Let T be a compact metric space and $\mathcal{C} = \mathcal{C}\{T, [0, +\infty)\}$ the Banach space of nonnegative continuous functions $x : T \rightarrow [0, +\infty)$ endowed with the uniform norm $\|x\| = \sup_{t \in T} |x(t)|$, $x \in \mathcal{C}$. Let $\mathcal{S} = \{x \in \mathcal{C}; \|x\| = 1\}$ be the unit sphere. Given any metric space \mathcal{X} , we denote by $\mathcal{B}(\mathcal{X})$ its Borel σ -algebra.

Definition 1. *A \mathcal{C} -valued random process X is said to be regularly varying with exponent $\alpha > 0$ and spectral probability measure σ on \mathcal{S} , noted shortly $X \in RV_{\alpha, \sigma}(\mathcal{C})$,*

if there exists a positive function $a(\cdot)$ such that $a(t) \rightarrow +\infty$ as $t \rightarrow +\infty$ and

$$t\mathbb{P}\{X/\|X\| \in B, \|X\| > ra(t)\} \rightarrow \sigma(B)r^{-\alpha}, \quad t \rightarrow \infty, \quad (1)$$

for all $r > 0$ and all $B \in \mathcal{B}(\mathcal{S})$ such that $\sigma(\partial B) = 0$ where ∂B denotes the boundary of B .

The exponent α and the spectral measure σ are uniquely determined while the function $a(\cdot)$ is unique up to asymptotic equivalence and regularly varying at infinity with exponent $1/\alpha$. Similarly to the univariate case, a convenient choice is

$$a(t) = \inf\{x \geq 0: \mathbb{P}(\|X\| \leq x) \leq 1 - 1/t\}, \quad (2)$$

and in the remainder of this paper we will always assume this choice.

We now introduce some technical backgrounds on function and measure spaces that are useful when using point processes. A first step is to introduce a suitable modification of the space \mathcal{C} in order to deal with points at infinity. In the univariate case, this is done by working with the space $(0, +\infty]$ instead of $[0, +\infty)$. Within a functional framework, we define $\mathcal{C}_0 = \mathcal{C} \setminus \{0\}$ and consider the complete separable metric space $\overline{\mathcal{C}}_0 = (0, +\infty] \times \mathcal{S}$ equipped with the metric

$$d\{(r_1, s_1), (r_2, s_2)\} = |1/r_1 - 1/r_2| + \|s_1 - s_2\|.$$

A set B is bounded in $\overline{\mathcal{C}}_0$ if and only if there exists some $\varepsilon > 0$ such that $B \subset [\varepsilon, +\infty] \times \mathcal{S}$. The polar decomposition $\mathcal{C}_0 \rightarrow (0, +\infty) \times \mathcal{S}$ given by $f \mapsto (\|f\|, f/\|f\|)$ is bijective and bi-continuous and allows to identify \mathcal{C}_0 and $(0, +\infty) \times \mathcal{S}$.

Definition 2. Let $M(\overline{\mathcal{C}}_0)$ be the set of Borel measures m on $\overline{\mathcal{C}}_0$ that are boundedly finite, i.e., such that $m(b) < \infty$, for all bounded sets $B \in \mathcal{B}(\overline{\mathcal{C}}_0)$.

A sequence $\{m_n\}_{n \geq 1}$ in $M(\overline{\mathcal{C}}_0)$ is said to converge to m in the \hat{w} -topology if

$$\int f dm_n \longrightarrow \int f dm, \quad n \rightarrow \infty,$$

for all bounded and continuous functions $f : \overline{\mathcal{C}}_0 \rightarrow \mathbb{R}$ with bounded support.

The notion of \hat{w} -convergence generalizes the notion of vague convergence and takes into account the fact that $\overline{\mathcal{C}}_0$ is not locally compact. The \hat{w} -topology defined by this notion of convergence ensures that $M(\overline{\mathcal{C}}_0)$ is a Polish space. The subspace $M_p(\overline{\mathcal{C}}_0)$ consisting of all boundedly finite point measures is a closed subset of $M(\overline{\mathcal{C}}_0)$ and is endowed with the induced \hat{w} -topology. It is the suitable space when working with point processes in functional extreme value theory.

In the following, we emphasize on the connections between regular variations, sample point measures, sample maxima and exceedances above high thresholds in a functional framework. Before generalizing the analogous of statements i)–v) in the univariate case to the functional framework, we first need some notations and to introduce the limiting objects that will appear in the functional extreme value theory.

Definition 3. For $\alpha > 0$ and σ a probability measure on \mathcal{S} , we define

- $m_{\alpha, \sigma}$ the unique measure on $\overline{\mathcal{C}}_0$ such that

$$m_{\alpha, \sigma}\{[r, +\infty) \times B\} = r^{-\alpha} \sigma(B), \quad r > 0, B \in \mathcal{B}(\mathcal{S});$$

- $\Pi_{\alpha, \sigma}$ a Poisson point measure on $\overline{\mathcal{C}}_0$ with intensity $m_{\alpha, \sigma}$;
- $M_{\alpha, \sigma}$ a continuous max-stable process on T with exponent measure $m_{\alpha, \sigma}$;
- $P_{\alpha, \sigma}$ a Pareto process with index $\alpha > 0$ and spectral measure σ , i.e.,

$$P_{\alpha, \sigma}(t) = P_\alpha Y(t), \quad t \in T$$

where P_α has an α -Pareto distribution, i.e., $\mathbb{P}(P_\alpha > r) = r^{-\alpha}$, $r \geq 1$, and is

independent of the continuous process Y defined on T and whose distribution is σ .

Note that $m_{\alpha,\sigma}$ is boundedly finite and homogeneous of order $-\alpha$, i.e.,

$$m_{\alpha,\sigma}(uA) = u^{-\alpha}m_{\alpha,\sigma}(A), \quad u > 0, \quad A \in \mathcal{B}(\overline{\mathcal{C}}_0) \text{ bounded.}$$

The Poisson point measure $\Pi_{\alpha,\sigma}$ can be seen as a random element of $M_p(\overline{\mathcal{C}}_0)$ and might be defined as follows. Let $\{\Gamma_i\}_{i \geq 1}$ be a Poisson point process on $(0, +\infty)$ with Lebesgue intensity and, independently, let $\{Y_i\}_{i \geq 1}$ be a sequence of independent processes with common distribution σ , then

$$\Pi_{\alpha,\sigma} = \sum_{i \geq 1} \delta_{\Gamma_i^{-1/\alpha} Y_i}$$

is a Poisson point process with intensity $m_{\alpha,\sigma}$. Similarly, the max-stable process

$$M_{\alpha,\sigma}(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Y_i(t), \quad t \in T,$$

is a max-stable process with exponent measure $m_{\alpha,\sigma}$.

Theorem 1. *Let X_1, X_2, \dots be independent copies of a continuous random process X on T . In item 3, we suppose furthermore that X has almost surely nonnegative sample paths. The following statements are equivalent*

1. $X \in RV_{\alpha,\sigma}(\mathcal{C})$;
2. the following \hat{w} -convergence holds in $M(\overline{\mathcal{C}}_0)$

$$t \mathbb{P}\{a(t)^{-1} X \in \cdot\} \xrightarrow{\hat{w}} m_{\alpha,\sigma}, \quad \text{as } t \rightarrow \infty;$$

3. the normalized point measure

$$N_n = \sum_{i=1}^n \delta_{X_i/a(n)}$$

converges weakly in $M_p(\overline{\mathcal{C}}_0)$ to the Poisson point measure $\Pi_{\alpha,\beta}$ as $n \rightarrow \infty$;

4. the normalized sample maximum

$$M_n(t) = a(n)^{-1} \max\{X_1(t), \dots, X_n(t)\}, \quad t \in T,$$

converges weakly in \mathcal{C} to the max-stable random process $M_{\alpha,\sigma}$ as $n \rightarrow \infty$;

5. the conditional distribution of normalized exceedances

$$\mathbb{P}(t^{-1}X \in A \mid \|X\| > t), \quad A \in \mathcal{B}(\mathcal{C}),$$

converges weakly in \mathcal{C} to the generalized Pareto process $P_{\alpha,\sigma}$ as $n \rightarrow \infty$.

Proof of Theorem 1. The equivalence $1 \Leftrightarrow 2$ is due to Hult and Lindskog [13]. The equivalence $1 \Leftrightarrow 3 \Leftrightarrow 4$ is proved in Davis and Mikosch [6]. The equivalence $4 \Leftrightarrow 5$ is a consequence of Theorem 4.1 in de Haan and Ferreira [8], where the the domain of attraction of general max-stable and Pareto processes are considered, and not only the simple case where the max-stable processes have standard Fréchet margins. For the convenience of the reader, we provide a direct and simple proof of the equivalence $1 \Leftrightarrow 5$. □

3 Peaks Over Thresholds and GPD processes

As explained in the introduction, the peaks over threshold approach amounts to consider the conditional distribution of a random field X given that $\ell(X) > u$, where $u > 0$ is the threshold level and $\ell: \mathcal{C} \rightarrow [0, +\infty)$ is a homogeneous measurable cost

functional associated to the threshold method.

3.1 GPD processes associated to the cost functional ℓ

The following theorem provides several equivalent characterizations of the ℓ -GPD process and generalizes Theorem 2.1 of de Haan and Ferreira [8].

Theorem 2. *Let W be a continuous stochastic process. The following three statements are equivalent:*

1. *Constructive approach: for all $t \in T$, $W(t) = PY(t)$ where*
 - 1a. *Y is a continuous stochastic process such that $\ell(Y) \equiv 1$;*
 - 1b. *P is a Pareto random variable with tail index $\alpha > 0$, i.e., $\mathbb{P}(P > u) = u^{-\alpha}$, $u > 1$;*
 - 1c. *Y and P are independent.*
2. *Homogeneity property:*
 - 2a. $\mathbb{P}\{\ell(W) > 1\} = 1$;
 - 2b. *For all $u \geq 1$ and measurable $A \subset \{f \in \mathcal{C}: \ell(f) \geq 1\}$,*

$$\mathbb{P}(W \in uA) = u^{-\alpha}\mathbb{P}(W \in A).$$

3. *Peaks over threshold stability:*
 - 3a. $\mathbb{P}\{\ell(W) > 1\} > 0$;
 - 3b. *For all $A \in \mathcal{B}(\mathcal{C})$ and all $u \geq 1$ such that $\mathbb{P}\{\ell(W) > u\} > 0$,*

$$\mathbb{P}\{u^{-1}W \in A \mid \ell(W) > u\} = \mathbb{P}(W \in A).$$

For the constructive approach 1. we have necessarily $P = \ell(W)$ and $Y = W/\ell(W)$ and the tail index α is the same in 1b. and 2b. Characterization 3. is more implicit and does not involve any tail index α .

Definition 4. The distribution σ of $W/\ell(W)$ is called the spectral measure. The process W is called a simple ℓ -Pareto process with tail index α and spectral measure σ and is denoted by $W \sim P_{\alpha,\sigma}^\ell$.

Proof of Theorem 2. We first prove that 1. \Rightarrow 2. Since $P = \ell(W)$, condition 2a. follows trivially from 1b. Consider the set

$$A_{v,B} = \{f \in \mathcal{C} : \ell(f) \geq v, f/\ell(f) \in B\}. \quad (3)$$

with $v \geq 1$ and $B \subset \{f \in \mathcal{C} : \ell(f) = 1\}$ measurable. Clearly,

$$\mathbb{P}(W \in A_{v,B}) = \mathbb{P}(P \geq v, Y \in B) = v^{-\alpha} \sigma(B).$$

Using the relation $uA_{v,B} = A_{uv,B}$, we obtain $\mathbb{P}(W \in uA_{v,B}) = u^{-\alpha} \mathbb{P}(W \in A_{v,B})$.

The sets of the form $A_{v,B}$ form a π -system and generate the σ -algebra of Borel sets $A \subset \{f \in \mathcal{C} : \ell(f) \geq 1\}$. Hence condition 2b. holds for all Borel set A .

We prove that 2. \Rightarrow 3. Let $A \subset \mathcal{C}$ be a Borel set. Using conditions 2a. and 2b., we obtain

$$\mathbb{P}\{u^{-1}W \in A, \ell(W) > u\} = u^{-\alpha} \mathbb{P}\{W \in A, \ell(W) > 1\} = u^{-\alpha} \mathbb{P}(W \in A).$$

When $A = \mathcal{C}$ we have $\mathbb{P}\{\ell(W) > u\} = u^{-\alpha} > 0$ and whence

$$\mathbb{P}\{u^{-1}W \in A \mid \ell(W) > u\} = \frac{\mathbb{P}\{u^{-1}W \in A, \ell(W) > u\}}{\mathbb{P}\{\ell(W) > u\}} = \mathbb{P}(W \in A).$$

It remains to check that 3. \Rightarrow 1. Condition 3b. with $A = \{f \in \mathcal{C} : \ell(f) > v\}$ gives

$$\mathbb{P}\{\ell(W) > uv\} = \mathbb{P}\{\ell(W) > u\} \mathbb{P}\{\ell(W) > v\}, \quad u, v \geq 1,$$

and hence the tail function $u \mapsto \bar{F}(u) = \mathbb{P}\{\ell(W) > u\}$ satisfies the functional equation

$$\bar{F}(uv) = \bar{F}(u)\bar{F}(v), \quad u, v \geq 1. \quad (4)$$

Condition 3a. gives the initial condition $\bar{F}(1) > 0$. Clearly (4) implies $\bar{F}(1) = \bar{F}(1)^2$ and the initial condition ensures that $\bar{F}(1) = 1$. We then prove that \bar{F} is positive on $[1, \infty)$. Since F is right continuous and $\bar{F}(\infty) = 0$, there exists some $u_0 > 1$ such that $\bar{F}(u_0) \in (0, 1)$. Using (4), we have for all $n \geq 1$, $\bar{F}(u_0^n) = \bar{F}(u_0)^n > 0$ and letting $u_0^n \rightarrow \infty$, \bar{F} must be positive since it must be non-increasing.

Any non-increasing positive solution of the functional equation (4) must be of the form $F(u) = u^{-\alpha}$ for some $\alpha > 0$. This proves that $P = \ell(W)$ satisfies condition 1b. Hence $\ell(W)$ is almost surely positive and $Y = W/\ell(W)$ satisfies condition 1a. It remains to prove that P and Y are independent. To this aim, we consider $B \in \mathcal{B}(\mathcal{C})$ and we set $A = \{f \in \mathcal{C}: f/\ell(f) \in B\}$. Condition 3b. ensures that for all $u \geq 1$,

$$\begin{aligned} \mathbb{P}(Y \in B, P > u) &= \mathbb{P}\{W \in A, \ell(W) > u\} = \mathbb{P}\{u^{-1}W \in A, \ell(W) > u\} \\ &= \mathbb{P}(W \in A)\mathbb{P}\{\ell(W) > u\} = \mathbb{P}(Y \in B)\mathbb{P}(P > u), \end{aligned}$$

and proves condition 1c. □

3.2 Exceedances over high threshold and Pareto processes

The following result relates convergence of normalized exceedances over a high threshold and Pareto processes. It relies on the peaks over threshold stability characterization of Pareto processes.

Proposition 1. *Assume that the cost functional $\ell : \mathcal{C} \rightarrow [0, +\infty)$ is continuous and let X be a stochastic process such that*

$$\mathbb{P}\{u^{-1}X \in \cdot \mid \ell(X) > u\} \longrightarrow \mathbb{P}(W \in \cdot), \quad u \rightarrow \infty,$$

weakly in \mathcal{C} .

Then either W is a simple ℓ -Pareto process or $\mathbb{P}\{\ell(W) = 1\} = 1$.

Proof of Proposition 1. Let $A = \{f \in \mathcal{C}: \ell(f) \geq 1\}$. Clearly, $\mathbb{P}\{u^{-1}X \in A \mid \ell(X) > u\} = 1$ for all $u > 0$. Furthermore, A is closed by the continuity of ℓ so that the Portmanteau theorem implies

$$\mathbb{P}(W \in A) \geq \limsup_{u \rightarrow +\infty} \mathbb{P}\{u^{-1}X \in A \mid \ell(X) > u\} = 1,$$

and hence $\ell(W) \geq 1$ almost surely.

We suppose that $\mathbb{P}\{\ell(W) = 1\} < 1$ and prove that W is a simple ℓ -Pareto process. Clearly, in this case, $\mathbb{P}\{\ell(W) > 1\} > 0$ and condition 3a. of Theorem 2 is satisfied. We prove that the limit W satisfies also the peak over threshold stability condition 3b. so that it must be a simple ℓ -Pareto process. Clearly, for all $u_1, u_2 \geq 1$ and all set $A_{v,B}$ of the form (3), we have

$$\begin{aligned} & \mathbb{P}\{u_1^{-1}u_2^{-1}X \in A_{v,B}, \ell(u_1^{-1}X) > v \mid \ell(X) > u_1\} \\ &= \mathbb{P}\{u_1^{-1}u_2^{-1}X \in A_{v,B} \mid \ell(X) > u_1u_2\} \mathbb{P}\{\ell(u_1^{-1}X) > u_2 \mid \ell(X) > u_1\}. \end{aligned}$$

As $u_1 \rightarrow \infty$ the weak convergence entails

$$\mathbb{P}\{u_2^{-1}W \in A_{v,B}, \ell(W) > u_2\} = \mathbb{P}(W \in A_{v,B}) \mathbb{P}\{\ell(W) > u_2\}, \quad (5)$$

provided $\mathbb{P}\{\ell(W) = v\} = 0$ and $\mathbb{P}\{W/\ell(W) \in \partial B\} = 0$. Indeed since ℓ is continuous, the boundary set of $\{f \in \mathcal{C}: \ell(f) > v\}$ is $\{f \in \mathcal{C}: \ell(f) = v\}$. Using regularity properties, (5) is extended to all $v \geq 1$ and all $A \in \mathcal{B}(\mathcal{C})$, and hence W satisfies condition 3b. \square

Theorem 3. *Suppose that $X \in RV_{\alpha,\sigma}(\mathcal{C})$. If ℓ is continuous at the origin and does*

not vanish σ -a.e., then

$$\mathbb{P}\{u^{-1}X \in \cdot \mid \ell(X) > u\} \longrightarrow P_{\sigma_\ell, \alpha}^\ell(\cdot), \quad u \rightarrow \infty,$$

weakly in \mathcal{C} and the spectral measure σ_ℓ is given by

$$\sigma_\ell(B) = \frac{1}{c_\ell} \int_{\mathcal{S}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}} \sigma(df), \quad B \in \mathcal{B}(\mathcal{C}), \quad (6)$$

with $c_\ell = \int_{\mathcal{S}} \ell(f)^\alpha \sigma(df)$.

Proof of Theorem 3. In order to use regular variations, we take $u = a(t)$ as the normalizing constant in Definition 1 and let $t \rightarrow \infty$. It is enough to prove that for all bounded continuous functional $F: \mathcal{C} \rightarrow \mathbb{R}$ we have

$$t\mathbb{E} [F\{a(t)^{-1}X\} 1_{\{\ell(X) > a(t)\}}] \longrightarrow c_\ell \mathbb{E}\{F(Y)\}, \quad t \rightarrow \infty, \quad (7)$$

with $Y \sim P_{\alpha, \sigma_\ell}^\ell$. Indeed, taking $F \equiv 1$ implies

$$t\mathbb{P}\{\ell(X) > a(t)\} \longrightarrow c_\ell, \quad t \rightarrow \infty.$$

Since ℓ is nonnegative and does not vanish σ -a.e., $c_\ell > 0$ and we have

$$\frac{\mathbb{E} [F(u^{-1}X) 1_{\{\ell(X) > u\}}]}{\mathbb{P}\{\ell(X) > u\}} \longrightarrow \mathbb{E}\{F(Y)\}, \quad u \rightarrow \infty,$$

and proves the required weak convergence. We prove (7). Using the homogeneity of ℓ , we have

$$t\mathbb{E} [F\{a(t)^{-1}X\} 1_{\{\ell(X) > a(t)\}}] = t\mathbb{E} [\tilde{F}\{a(t)^{-1}X\}], \quad (8)$$

with $\tilde{F}(f) = F(f) 1_{\{\ell(f) > 1\}}$. According to Theorem 1, $X \in \text{RV}_{\alpha, \sigma}(\mathcal{C})$ implies the

\hat{w} -convergence $t\mathbb{P}\{a(t)^{-1}X \in \cdot\} \rightarrow m_{\alpha,\beta}(\cdot)$ in $\overline{\mathcal{C}_0}$ and therefore we have

$$t\mathbb{E}\left[\tilde{F}\{a(t)^{-1}X\}\right] \longrightarrow \int_{\overline{\mathcal{C}_0}} \tilde{F}(f)m_{\alpha,\sigma}(df), \quad (9)$$

provided that \tilde{F} has a bounded support in $\overline{\mathcal{C}_0}$ and is continuous $m_{\alpha,\sigma}$ -a.e. We will check these conditions later. The right hand side of (9) is equal to

$$\int_0^\infty \int_{\mathcal{S}} \tilde{F}(rf) \alpha r^{-\alpha-1} dr \sigma(df) = \int_0^\infty \int_{\mathcal{S}} F(rf) 1_{\{r\ell(f)>1\}} \alpha r^{-\alpha-1} dr \sigma(df). \quad (10)$$

On the other hand, the right hand side of (7) can be computed using a simple change of variable

$$\begin{aligned} \int_1^\infty \int_{\mathcal{S}^\ell} F(rf) \alpha r^{-\alpha-1} dr c_\ell \sigma_\ell(df) &= \int_0^\infty \int_{\mathcal{S}} 1_{\{r>1\}} F(rf/\ell(f)) \ell(\alpha)^\alpha \alpha r^{-\alpha-1} dr \sigma(df) \\ &= \int_0^\infty \int_{\mathcal{S}} 1_{\{r\ell(f)>1\}} F(rf) \alpha r^{-\alpha-1} dr \sigma(df). \end{aligned} \quad (11)$$

Equations (8)–(11) imply (7) and it remains to prove that \tilde{F} has a bounded support in $\overline{\mathcal{C}_0}$ and is continuous $m_{\alpha,\sigma}$ -a.e. The continuity and homogeneity of ℓ implies that there exists some $M > 0$ such that $\ell(f) \leq M\|f\|$ for all $f \in \mathcal{C}$. Hence $\ell(f) > 1$ implies $\|f\| > M^{-1}$ and the support of \tilde{F} is included in $[M^{-1}, +\infty] \times \mathcal{S}$ and is bounded in $\overline{\mathcal{C}_0}$. Furthermore since F is continuous, $f \mapsto \tilde{F}(f) = F(f)1_{\{\ell(f)>1\}}$ is continuous at every point f such that $\ell(f) \neq 1$. Finally we have that

$$m_{\alpha,\sigma}(\{\ell(f) = 1\}) = \int_0^\infty 1_{\{r\ell(f)=1\}} \alpha r^{-\alpha-1} dr \sigma(df) = 0,$$

and \tilde{F} is continuous $m_{\alpha,\sigma}$ -a.e. □

When different functionals ℓ and ℓ' are involved, the corresponding spectral measures σ_ℓ and $\sigma_{\ell'}$ defined by (6) are linked by a simple relation.

Proposition 2. *Let ℓ and ℓ' be homogeneous measurable functionals $\mathcal{C} \rightarrow [0, +\infty)$*

and suppose that $\ell'(f) > 0$ σ_ℓ -a.e. Then,

$$\sigma_\ell(B) = \frac{\int_{\mathcal{B}(\mathcal{C})} \ell(f)^\alpha \mathbf{1}_{\{f/\ell(f) \in B\}} \sigma_{\ell'}(df)}{\int_{\mathcal{B}(\mathcal{C})} \ell(f)^\alpha \sigma_{\ell'}(df)}, \quad B \in \mathcal{B}(\mathcal{C}). \quad (12)$$

As a direct consequence, if $\ell(f) > 0$ σ -a.e., then (6) can be inverted and we have

$$\sigma(B) = c_\ell \int_{\mathcal{S}^\ell} \|f\|^\alpha \mathbf{1}_{\{f/\|f\| \in B\}} \sigma_\ell(df), \quad B \in \mathcal{B}(\mathcal{S}). \quad (13)$$

Proof of Proposition 2. Using the definition of σ_ℓ and $\sigma_{\ell'}$, we have for $B \in \mathcal{B}(\mathcal{C})$,

$$\begin{aligned} \int_{\mathcal{B}(\mathcal{C})} \ell(f)^\alpha \mathbf{1}_{\{f/\ell(f) \in B\}} \sigma_{\ell'}(df) &= \frac{1}{c_{\ell'}} \int_{\mathcal{B}(\mathcal{C})} \ell'(f)^\alpha \{\ell(f)/\ell'(f)\}^\alpha \mathbf{1}_{\{f/\ell(f) \in B\}} \sigma(df) \\ &= \frac{c_\ell}{c_{\ell'}} \sigma_\ell(B), \end{aligned}$$

where we used in the last equality the fact that $\ell'(f) > 0$ σ_ℓ -a.e. Taking $B = \mathcal{C}$, we get

$$c_\ell = c_{\ell'} \int \ell(f)^\alpha \sigma_{\ell'}(df)$$

and (12) follows easily. □

4 Estimation of spectral measures

Under the assumptions of Theorem 3, we consider a natural non-parametric estimator of the spectral measure σ_ℓ in (6) associated to the regularly varying random field $X \in \text{RV}_{\alpha, \sigma}$. It is based on independent copies X_1, X_2, \dots of X , and especially on exceedances over large thresholds, i.e., such that $\ell(X_i) > u_n$ for some large threshold level u_n .

Proposition 3. *Suppose that $X \in \text{RV}_{\alpha, \sigma}$ and that ℓ is continuous at the origin and does not vanish σ -a.e. Consider a sequence $u_n > 0$ such that $u_n \rightarrow \infty$ and*

$u_n/a(n) \rightarrow 0$, with $a(\cdot)$ given by (2). Then

$$\hat{\sigma}_{\ell,n} = \frac{\sum_{i=1}^n 1_{\{\ell(X_i) > u_n\}} \delta_{X_i/\ell(X_i)}}{\sum_{i=1}^n 1_{\{\ell(X_i) > u_n\}}}, \quad n \geq 1,$$

is a consistent estimator of the spectral measure σ_ℓ in the sense that $\hat{\sigma}_{\ell,n}(B)$ converges in probability as $n \rightarrow +\infty$ to $\sigma_\ell(B)$ for all $B \in \mathcal{B}(\mathcal{C})$ such that $\sigma_\ell(\partial B) = 0$.

In some applications, it may happen that the observations are obtained by thresholding with respect to a functional ℓ' different from the functional ℓ of interest. Propositions 2 and 3 suggest the following generalized estimator.

Proposition 4. *Suppose that $X \in \text{RV}_{\alpha,\sigma}$. Let ℓ and ℓ' be homogeneous functionals $\mathcal{C} \rightarrow [0, +\infty)$ such that*

- ℓ' is continuous at the origin and does not vanish σ -a.e.
- ℓ is continuous and satisfies $\ell \leq M\ell'$ for some $M > 0$.

Consider a sequence $u_n > 0$ such that $u_n \rightarrow \infty$ and $u_n/a_n \rightarrow 0$ with $a(\cdot)$ given by (2). Then

$$\tilde{\sigma}_{\ell,n} = \frac{\sum_{i=1}^n \{\ell(X_i)/\ell'(X_i)\}^\alpha 1_{\{\ell'(X_i) > u_n\}} \delta_{\{X_i/\ell(X_i)\}}}{\sum_{i=1}^n \{\ell(X_i)/\ell'(X_i)\}^\alpha 1_{\{\ell'(X_i) > u_n\}}}.$$

is a consistent estimator of σ_ℓ .

Proof of Proposition 3. Let $N_n = \sum_{i=1}^n 1_{\{\ell(X_i) > u_n\}}$ be the number of observations above threshold u_n in the sample X_1, \dots, X_n . The estimator $\hat{\sigma}_{\ell,n}$ is well defined as soon as $N_n > 0$. If $N_n = 0$, define $\hat{\sigma}_{\ell,n} = \sigma_0$ with σ_0 an arbitrary probability measure. The choice of σ_0 is irrelevant from an asymptotic point of view since we will see that $\mathbb{P}(N_n = 0) \rightarrow 0$ as $n \rightarrow \infty$.

Clearly N_n has a binomial distribution with parameters n and $p_n = \mathbb{P}\{\ell(X) > u_n\}$. In particular, N_n has mean np_n and variance $np_n(1 - p_n)$. The conditions $u_n \rightarrow \infty$ and $u_n/a_n \rightarrow 0$ imply $p_n \rightarrow 0$ and $np_n \rightarrow \infty$. Since $N_n/(np_n)$ has mean 1 and its variance goes to 0 as $n \rightarrow \infty$, $N_n/(np_n)$ converges in probability to 1.

Let $B \in \mathcal{B}(\mathcal{C})$ such that $\sigma_\ell(\partial B) = 0$ and define $p_{n,B} = \mathbb{P}\{\ell(X) > u_n, X/\ell(X) \in B\}$. The normalized sum

$$\frac{1}{np_n} \sum_{i=1}^n 1_{\{\ell(X_i) > u_n, X_i/\ell(X_i) \in B\}} \quad (14)$$

has expectation $p_{n,B}/p_n$ and variance $p_{n,B}(1 - p_{n,B})/(np_n^2)$. Theorem 3 combined with the condition $\sigma_\ell(\partial B) = 0$ yields

$$\frac{p_{n,B}}{p_n} = \mathbb{P}\{X/\ell(X) \in B \mid \ell(X) > u_n\} \longrightarrow \sigma_\ell(B), \quad n \rightarrow \infty.$$

Since

$$\frac{p_{n,B}(1 - p_{n,B})}{np_n^2} \sim \frac{\sigma_\ell(B)}{np_n} \longrightarrow 0,$$

(14) converges in probability to $\sigma_\ell(B)$ as $n \rightarrow \infty$.

Finally, the expression

$$\hat{\sigma}_{\ell,n}(B) = \frac{np_n}{N_n} \frac{1}{np_n} \left[\sum_{i=1}^n 1_{\{\ell(X_i) > v_n, X_i/\ell(X_i) \in B\}} \right] 1_{\{N_n > 0\}} + \sigma_0 1_{\{N_n = 0\}},$$

and the convergences in probability mentioned above combined with Slutsky's lemma imply $\hat{\sigma}_{\ell,n}(B) \rightarrow \sigma_\ell(B)$ in probability and proves Proposition 3. \square

Proof of Proposition 4. According to Proposition 2,

$$\sigma_\ell(B) = \frac{\int \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}} \sigma_{\ell'}(df)}{\int \ell(f)^\alpha \sigma_{\ell'}(df)}, \quad B \in \mathcal{B}(\mathcal{C}). \quad (15)$$

The estimator $\tilde{\sigma}_{\ell,n}$ is obtained by replacing $\sigma_{\ell'}$ in this expression by the non-parametric estimator $\hat{\sigma}_{\ell',n}$ from Proposition 3:

$$\tilde{\sigma}_{\ell,n}(B) = \frac{\int \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}} \hat{\sigma}_{\ell',n}(df)}{\int \ell(f)^\alpha \hat{\sigma}_{\ell',n}(df)}, \quad B \in \mathcal{B}(\mathcal{C}). \quad (16)$$

As a consequence of Proposition 3, the probability measures $\hat{\sigma}_{\ell',n}$ converge in probabil-

ity as $n \rightarrow \infty$ to $\sigma_{\ell'}$ in the space of probability measures on \mathcal{C} endowed with the metric of weak convergence. This entails the convergence in probability $\int F(f)\hat{\sigma}_{\ell',n}(df) \rightarrow \int F(f)\sigma_{\ell'}(df)$ for all functional F that are bounded and continuous $\sigma_{\ell'}$ -a.e.

Let $B \in \mathcal{B}$ be such that $\sigma_{\ell}(\partial B) = 0$ and consider the particular choice $F_B(f) = \ell(f)^{\alpha}1_{\{f/\ell(f) \in B\}}$. The condition $\ell \leq M\ell'$ entails $F_B(f) \leq M^{\alpha}$ for all f such that $\ell'(f) = 1$. Since ℓ is continuous, F_B is continuous except at points f such that $f/\ell(f) \in \partial B$. It is easily checked that the condition $\sigma_{\ell}(\partial B) = 0$ implies $\sigma_{\ell'}(\{f/\ell(f) \in \partial B\}) = 0$ so that F_B is continuous $\sigma_{\ell'}$ -a.e. Hence, we get

$$\int \ell(f)^{\alpha}1_{\{f/\ell(f) \in B\}} \hat{\sigma}_{\ell',n}(df) \xrightarrow{\mathbb{P}} \int \ell(f)^{\alpha}1_{\{f/\ell(f) \in B\}} \hat{\sigma}_{\ell'}(df), \quad n \rightarrow \infty$$

and similarly

$$\int \ell(f)^{\alpha} \hat{\sigma}_{\ell',n}(df) \xrightarrow{\mathbb{P}} \int \ell(f)^{\alpha}1_{\{f/\ell(f) \in B\}} \hat{\sigma}_{\ell'}(df), \quad n \rightarrow \infty.$$

Then (15)–(16) and Slutsky's Lemma imply that $\tilde{\sigma}_{\ell,n}(B) \rightarrow \sigma_{\ell}(B)$ in probability. \square

5 Examples of regularly varying random fields

We review some standard examples of regularly varying random fields, following section 4.1 in Davis and Mikosh [6].

- **Simple multiplicative processes.**

Consider the \mathcal{C} -valued random process $X(t) = \eta Y(t)$, $t \in T$, where η and Y are independent and such that

- η is a non-negative regularly varying random variable with index $\alpha > 0$,
- Y is a sample continuous random field on T .

Assume that one of the following conditions is satisfied

- i) $\mathbb{E}(\|Y\|^{\alpha+\delta}) < \infty$ for some $\delta > 0$;

ii) $\mathbb{E}(\|Y\|^\alpha) < \infty$ and $\mathbb{P}(\eta > x) \sim Cx^{-\alpha}$ for some $C > 0$ as $x \rightarrow \infty$.

Then, according to section 4.1 in [6], the process X is regularly varying on \mathcal{C} with index α and spectral measure given by

$$\sigma(A) = \frac{\mathbb{E}[\|Y\|^\alpha 1_{\{Y/\|Y\| \in A\}}]}{\mathbb{E}[\|Y\|^\alpha]}, \quad A \in \mathcal{B}(\mathcal{S}). \quad (17)$$

• **Symmetric α -stable processes.**

Any sample continuous symmetric α -stable processes on T can be represented as a LePage series

$$X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} Y_i(t), \quad t \in T, \quad (18)$$

where $\alpha \in (0, 2)$ and the sequences $\{\Gamma_i\}_{i \geq 1}$ and $\{Y_i\}_{i \geq 1}$ are independent and such that

- the sequence $\{\Gamma_i\}_{i \geq 1}$ is the non decreasing enumeration of the points of a Poisson Point Process with Lebesgue intensity on $(0, +\infty)$,
- the Y_i are independent copies of a \mathcal{C} -valued symmetric random fields Y .

Such random fields are regularly varying on \mathcal{C} with index α and spectral measure σ given by (17). The normalizing sequence $\{a_n\}_{n \geq 1}$ satisfies

$$a_n \sim \mathbb{E}(\|Y\|^\alpha)^{1/\alpha} n^{1/\alpha}, \quad n \rightarrow \infty. \quad (19)$$

For more details on symmetric α -stable random fields, the reader should refer to Samorodnitsky and Taqqu [18] or Ledoux and Talagrand [15].

• **Max-stable processes.**

Any sample continuous simple max-stable processes on T with α -Fréchet margins, $\alpha > 0$, can be represented as

$$X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Y_i(t), \quad t \in T, \quad (20)$$

with the sequences $\{\Gamma_i\}_{i \geq 1}$ and $\{Y_i\}_{i \geq 1}$ as in (18). Conversely, the infinite maximum (20) converges almost surely in \mathcal{C} if and only if $\mathbb{E}(\|Y\|^\alpha) < \infty$ and for such cases the limit X is a sample continuous max-stable process on \mathcal{C} with α -Fréchet margins

$$\mathbb{P}\{X(t) \leq x\} = \exp[-\mathbb{E}\{Y(t)^\alpha\}x^{-\alpha}], \quad x > 0, t \in T.$$

Max-stable random fields are regularly varying on \mathcal{C} with index α and spectral measure σ given by (17) and the renormalising sequence $\{a_n\}_{n \geq 1}$ satisfies (19).

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